第一章、马氏链

§1.1 定义与例子

1.定义

- 离散型分布/随机变量.
- S: 可数集, 取值/位置/状态空间.

  $i \in S$: 值/位置/状态.
- 分布(列), 色子: $\{p_i,\ i \in S\}$.
- 随机变量, 粒子: $P(X = i) = p_i,\ \forall i$.
- 马氏链(模型)描述一个在S 上运动的粒子.
- 离散时间: $n = 0, 1, 2, \ldots,\ n \in \mathbb{Z}_+$.
- 粒子的位置: 一列取值于S 的随机变量

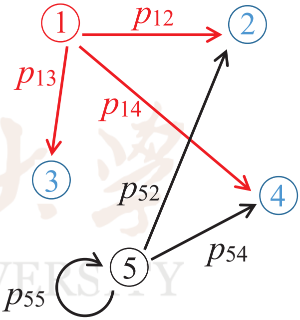
$$X_0, X_1, X_2, \ldots, \quad 也记为\ \{X_n\}.$$

① ② ③ ④ ⑤

- 运动机制/概率转移图：每个位置i 上放置色子$\{p_{ij}, j \in S\}$.

  (1) $p_{ij} \geqslant 0, \forall i, j$;

  (2) $\sum_j p_{ij} = 1, \forall i$.



- 转移(概率)矩阵：

$$\mathbf{P} = (p_{ij})_{i,j\in S} = (p_{ij})_{S\times S} = (p_{ij}).$$

- $\{X_n\}$ 是(时齐)马氏链指：

  $\forall n \geqslant 0$; $\forall i, j, i_0, \ldots, i_{n-1} \in S$,

  若$P(A, C) > 0$, 则

$$P\left(X_{n+1} = j \middle| X_n = i, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}\right) = p_{ij}.$$

- 马氏链= 发展机制: $\mathbf{P} = (p_{ij})_{S\times S}$.
- 再配上初分布μ: $\mu_i = P(X_0 = i)$, 得到有限维轨道分布:

$$
\begin{aligned}
&P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) \\
=&P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) \\
&\times P(X_n = i_n | X_0 = i_0, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i_{n-1}) \\
=&P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \times p_{i_{n-1} i_n} = \cdots \\
=&\mu_{i_0} p_{i_0 i_1} \ldots p_{i_{n-2} i_{n-1}} p_{i_{n-1} i_n}.
\end{aligned}
$$

- 或者,

$$
\begin{aligned}
P(X_0 = i_0) =& \mu_{i_0}, \\
P(X_0 = i_0, X_1 = i_1) =& \mu_{i_0} p_{i_0 i_1}, \\
P(X_0 = i_0, X_1 = i_1, X_2 = i_2) =& \mu_{i_0} p_{i_0 i_1} p_{i_1 i_2}, \quad \cdots\cdots
\end{aligned}
$$

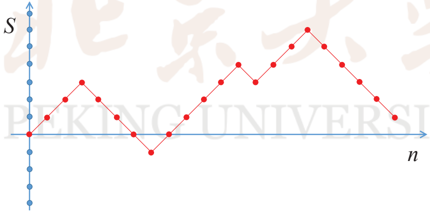
2. 例子

例1.1.1. 一维随机游动. $S=\mathbb{Z}$.

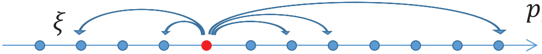
- 简单随机游动: $p_{i,i+1}=p_{i,i-1}=\frac{1}{2}$.

紧邻随机游动: $p_{i,i+1}=1-p_{i,i-1}=p$.

- 轨道图:



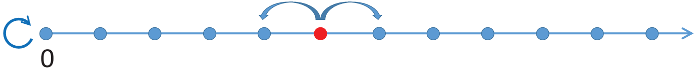
- 一维随机游动: $\xi_1,\xi_2,\ldots$ i.i.d., $S_n=\xi_1+\ldots+\xi_{n-1}+\xi_n$.



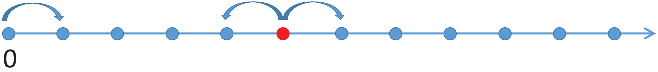
$p_{i,i+k}=P(\xi=k)$

相关模型:

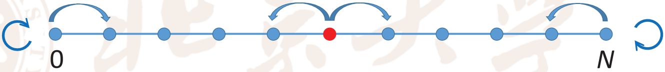
- 吸收壁



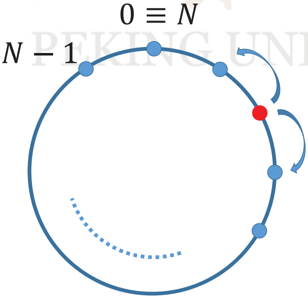
- 反射壁



- 粘滞边界/区间上

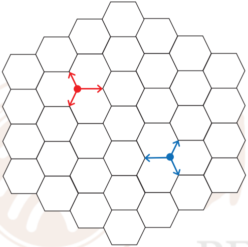


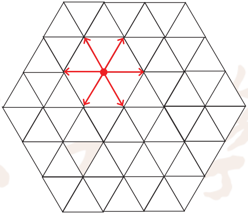
- 离散圆圈上



例1.1.11. 图上的(简单)随机游动.

- 图$G = (V, E)$.





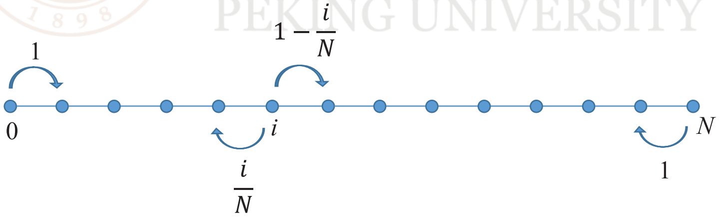
- 位置空间: $S = V$.

$$p_{i,j} = \begin{cases} \frac{1}{d_i}, & \text{若} j \sim i, \\ 0, & \text{否则}. \end{cases}$$

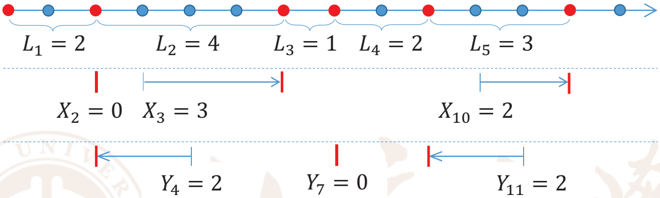
例1.1.8. Ehrenfest模型. N 个球, 两个纸箱A, B.
每次独立地随机选一个球, 把它换到另一个纸箱中.

- X_n: n 次操作后纸箱A 的状态, 即, 其中球的个数.
  则$\{X_n\}$ 是马氏链.
- $S=\{0,1,\dots,N\}$,

$$p_{i,i-1}=\frac{i}{N},\quad p_{i,i+1}=1-\frac{i}{N}.$$



例1.1.9 & 1.1.10. $L_1, L_2, \ldots$ i.i.d., 取正整数.



- X_n: 灯泡的余寿(更新过程). $S = \mathbb{Z}_+$.

$$p_{i,i-1} = 1,\ \forall i \geqslant 1, \qquad p_{0,i} = P(L = i+1),\ \forall i \geqslant 0.$$

- Y_n: 灯泡已经使用的时间(老化过程). $S = \mathbb{Z}_+$.

$$p_{i,i+1} = 1 - p_{i,0},\ \forall i \geqslant 0.$$

3. 性质

- $P(X_{n+1}=j|X_n=i,\vec{Z}=\vec{i})=p_{ij}$,
  $\vec{Z}=(X_0,\ldots,X_{n-1})$, $\vec{i}=(i_0,\ldots,i_{n-1})$.
- 由条件概率的定义$P_A(B)=\frac{P(AB)}{P(A)}$知,

$$P(B|A,C)=\frac{P(A,B,C)}{P(A,C)}=\frac{P_A(B,C)}{P_A(C)}=P_A(B|C).$$

- 翻译马氏链定义: $P_A(X_{n+1}=j|\vec{Z}=\vec{i})=p_{ij}$. (不依赖于$\vec{i}$.)
- 在P_A 下, X_{n+1}, $\vec{Z}$ 相互独立:

$$P_A(X_{n+1}=j,\vec{Z}=\vec{i})=p_{ij}\times P_A(\vec{Z}=\vec{i}).$$

对$\vec{i}$ 求和知$P_A(X_{n+1}=j)=p_{ij}$, 再代入上式即可.

- 命题1.1.12.(马氏性).

  在已知现在(的状态)的条件下, 将来与过去相互独立.

- 现在: 时刻n, 变量X_n;　　已知: $X_n = i$.

  将来: $\vec{Y} = (X_{n+1}, \ldots, X_{n+m})$;　过去: $\vec{Z} = (X_0, \ldots, X_{n-1})$.

- 已知$A = \{X_n = i\}$,

  将来事件$B = \{\vec{Y} = (j_1, \ldots, j_m) =: \vec{j}\}$;

  过去事件$C = \{\vec{Z} = (i_0, \ldots, i_{n-1}) =: \vec{i}\}$.

- 命题1.1.12. $\forall n, m$, $\forall \vec{i}$, $\forall \vec{j}$, 都有

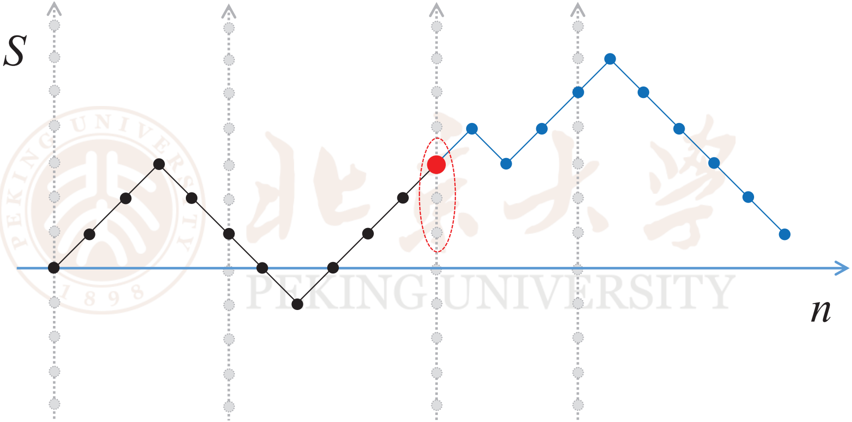
$$P_A(B|C) = p_{ij_1} \cdots p_{j_{m-1}j_m}.$$

命题1.1.12.的证明:

- $A=\{X_n=i\}$, $B=\{\vec{Y}=\vec{j}\}$, $C=\{\vec{Z}=\vec{i}\}$.

$$P_A(B|C)=P(B|A,C)=\frac{P(C,A,B)}{P(A,C)}$$
$$=\frac{\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i}\times p_{ij_1}\cdots p_{j_{m-1}j_m}}{\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i}}$$
$$=p_{ij_1}\cdots p_{j_{m-1}j_m}.$$

- 注1. 上式右边不依赖于$\vec{i}$. 因此, 在P_A 下, $\vec{Y}$ 与$\vec{Z}$ 相互独立.
- 注2: 特别地, 补充定义$Y_0=X_n$,
  则在P_A 与$P_{A,C}$ 下, $\{Y_m\}$ 是从i 出发的马氏链.
- 注3. 在P_A 下, $\forall 1\leqslant m_1<\ldots<m_r$; $0\leqslant n_1<\ldots<n_s<n$,
  $(X_{n+m_1},\ldots,X_{n+m_r})$ 与$(X_{n_1},\ldots,X_{n_s})$ 相互独立.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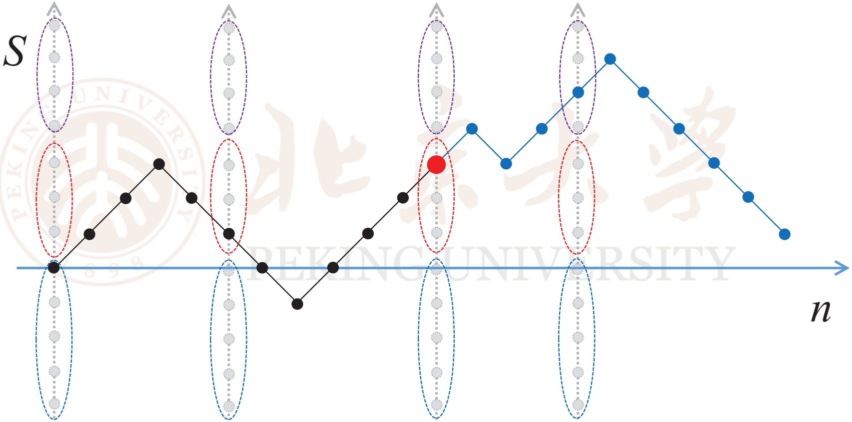
n

习题10.



- n 步转移概率:

$$p_{ij}^{(n)} := P(X_{m+n} = j | X_m = i).$$

- $\forall r \geqslant 1$, $\forall 0 < n_1 < \cdots < n_r$, $\forall i_0, i_1, \cdots, i_r$,

$$\begin{aligned}
&P(X_0 = i_0, X_{n_1} = i_1, \ldots, X_{n_r} = i_r) \\
=&P(X_0 = i_0, X_{n_1} = i_1, \ldots, X_{n_{r-1}} = i_{r-1}) \\
&\times P(X_{n_r} = i_r | X_0 = i_0, X_{n_1} = i_1, \ldots, X_{n_{r-1}} = i_{r-1}) \\
=&P(X_0 = i_0, \ldots, X_{n_{r-1}} = i_{r-1}) P(X_{n_r} = i_r | X_{n_{r-1}} = i_{r-1}) \\
=&\mu_{i_0} p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2 - n_1)} \ldots p_{i_{r-1} i_r}^{(n_r - n_{r-1})}.
\end{aligned}$$

- n 步转移矩阵: $(p_{ij}^{(n)})_{S \times S} =: \mathbf{P}^{(n)}$.

- Chapman-Kolmogorov等式:

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)}.$$

- 概率角度: 由全概公式,

$$\begin{aligned} & P(X_{n+m} = j | X_0 = i) \\ = & \sum_k P(X_{n+m} = j, X_n = k | X_0 = i) \\ = & \sum_k P(X_n = k | X_0 = i) P(X_{n+m} = j | X_n = k, X_0 = i). \end{aligned}$$

- 矩阵角度: $\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \times \mathbf{P}^{(m)}$.

  由归纳法知, $\mathbf{P}^{(n)} = \mathbf{P}^n$.

4. 补充.

- 命题1.1.12.(马氏性).

  在已知现在(的状态)的条件下, 将来与过去相互独立.

- 思考: 假设$U_0, U_1, U_2, \cdots$ i.i.d., $U_0 \sim U(0,1)$.

  如何用$U_0, U_1, U_2, \cdots$ 构造马氏链?

- 马氏链模型对应的随机试验.
- 样本/轨道:

$$\Omega := \{\omega = (\omega_0, \omega_1, \omega_2, \ldots) : \omega_n \in S, \forall n \geqslant 0\}.$$

- (坐标)过程:

$$X_0(\omega) := \omega_0, \ X_1(\omega) := \omega_1, \ X_2(\omega) := \omega_2, \ldots$$

- σ-代数:

$$\mathcal{F} := \sigma(\mathcal{G}) = \sigma(\mathcal{C}),$$

$$\mathcal{G} : \{X_n = i\}, \ \forall n \geqslant 0, \ \forall i \in S,$$

$$\mathcal{C} : C_{i_0 i_1 \ldots i_n} = \{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} \quad (\text{柱集}).$$

- 概率: ∃! P 使得

$$
\begin{aligned}
&P(X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n) \\
=&P(C_{i_0 i_1 \dots i_n}) = \mu_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}, \quad \forall n \geqslant 0, \forall i_0, i_1, \cdots, i_n \in S.
\end{aligned}
$$

- P 也是$\{X_n\}$ 的"联合分布", 被称为轨道分布.
- 随机过程与轨道分布.

§1.2 不变分布与可逆分布

1.不变分布

- 初分布μ & 转移机制$\mathbf{P}=(p_{ij})$.

$$P_\mu(X_0=i_0,X_1=i_1,\ldots,X_n=i_n)=\mu_{i_0}p_{i_0i_1}\ldots p_{i_{n-1}i_n}.$$

当$\mu_i=1$ 时, P_μ 简记为P_i.

- P_μ 与P_i:

$$P_\mu(A)=\sum_i \mu_i P_i(A),\quad P_i(A)=P_\mu(A|X_0=i).$$

- X_1 的分布:

$$P_\mu(X_1=j)=\sum_i P_\mu(X_0=i,X_1=j)=\sum_i \mu_i p_{ij}=(\mu\mathbf{P})_j.$$

- X_2 的分布: $(\mu\mathbf{P})\mathbf{P}=\mu\mathbf{P}^2$, $\cdots$. 或, X_n 的分布:

$$P_\mu(X_n=j)=\sum_i \mu_i p_{ij}^{(n)}=(\mu\mathbf{P}^n)_j.$$

- 不变分布/测度(定义1.2.1): 若π 是分布/测度, 且

$$\sum_i \pi_i p_{ij} = \pi_j, \quad \forall j,$$

则称π 为不变分布/测度.

- $\mathbf{P}: \mu \mapsto \mu\mathbf{P}$. 不变分布= 线性变换$\mathbf{P}$ 的(左)不动点.
- 特征向量: $\pi\mathbf{P} = \pi$, $\mathbf{P1} = \mathbf{1}$. (Perron-Frobenius 定理)
- $\mathbf{P}: f \mapsto \mathbf{P}f$,

$$(\mathbf{P}f)(i) = \sum_j p_{ij} f(j) = E_i f(X_1).$$

- 右不动点被称为调和函数.

- $X_0 \sim \pi$ 则$X_n \sim \pi$, $\forall n \geqslant 0$. 不变分布= 系统位于稳态.
- 若

$$(X_0, X_1, \ldots, X_m) \stackrel{d}{=} (X_n, X_{n+m}, \ldots, X_{n+m}), \ \forall n, m \geqslant 0,$$

则称$\{X_n\}$ 平稳. 不变分布= 产生平稳过程.

- 假设$\{X_n\}$ 平稳, $f: S^\infty \to S'$. 令$Y_n = f(X_n, X_{n+1}, \cdots)$, 则$\{Y_n\}$ 平稳.

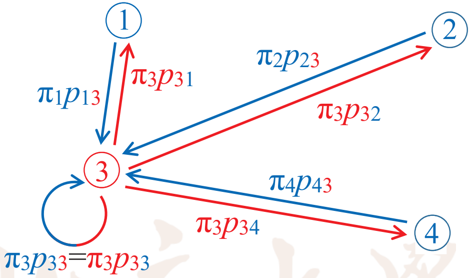
- 概率流: $\forall i$,

  $\sum_j \pi_j p_{ji} = \pi_i$,

  $\sum_{j\neq i} \pi_j p_{ji}$

  $= \pi_i \sum_{j\neq i} p_{ij}$

  $= \pi_i(1 - p_{ii})$

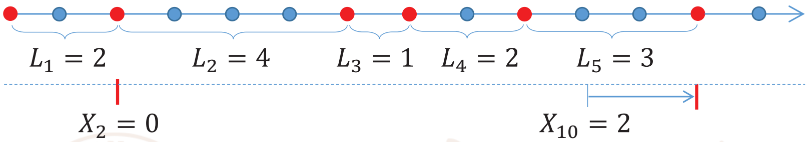


- 不变分布= 每个状态的总收与总支相抵.
- 不变分布= 每个状态组的总收与总支相抵.

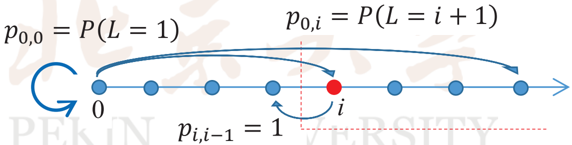
$$\sum_{i\in A}\sum_{j}\pi_j p_{ji} = \sum_{j\in A}\pi_i = \sum_{i\in A}\sum_{j}\pi_i p_{ij},$$

$$\sum_{j\notin A, i\in A}\pi_j p_{ji} + \sum_{i,j\in A}\pi_j p_{ji} = \sum_{i\in A, j\notin A}\pi_i p_{ij} + \sum_{i,j\in A}\pi_i p_{ij}.$$

例1.2.4. 更新过程. $L_1, L_2, \ldots$ i.i.d., $\geqslant 1$. 求$\{Y_n\}$的不变分布.

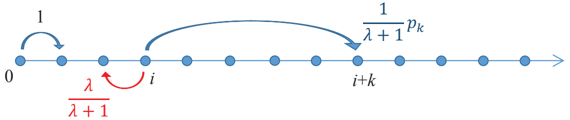


- 概率转移图:

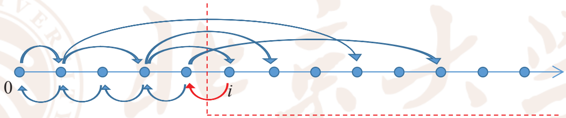


- 取$A = \{i, i+1, \ldots\}$, 则$\pi_0 P(L > i) = \pi_i$.
- $\sum_i \pi_i = \pi_0(1 + P(L > 1) + P(L > 2) + \ldots) = \pi_0 EL$.
- 不变分布存在iff $EL < \infty$, 此时$\pi_i = \frac{1}{EL} P(L > i)$, $\forall i$.

例1.2.5. $S=\mathbb{Z}_+$, $1=\sum_{k=1}^{\infty}p_k<m:=\sum_{k=1}^{\infty}kp_k<\lambda$.



- 取$A_i=\{i,i+1,i+2,\dots\}$.



- 当$i=1$ 时: $\pi_1\frac{\lambda}{\lambda+1}=\pi_0$.
- 当$i\geqslant 2$ 时: $\pi_i\frac{\lambda}{\lambda+1}=\sum_{j=1}^{i-1}\pi_j\frac{f_{i-j}}{\lambda+1}$, 其中$f_r=p_r+p_{r+1}+\dots$.
- $\sum_{i=j+1}^{\infty}f_{i-j}=f_1+f_2+\cdots=m$, 故

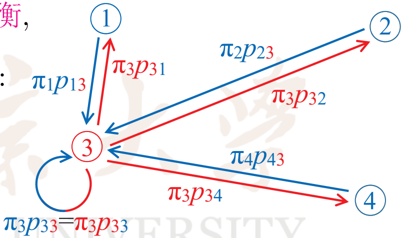
$$(1-\pi_0)\frac{\lambda}{\lambda+1}\times 1=\pi_0\times 1+(1-\pi_0)\frac{1}{\lambda+1}\times m.$$

2. 可逆分布

- 细致平衡条件:

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad \forall i, j.$$

- 细致平衡= 每对状态的收支平衡,
- 可逆分布/配称测度(定义1.2.7): 满足细致平衡的分布/测度.
- 若存在可逆分布/配称测度, 则称**P** 为可逆的/可配称的.



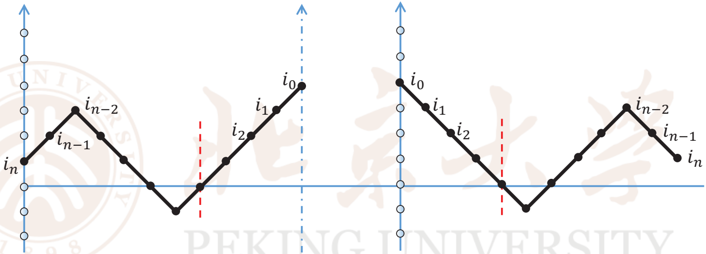
- 配称⇒ 不变.　可逆分布iff 配称& 归一.
- 可配称的含义:

$$q_{ij} = \sqrt{\frac{\pi_i}{\pi_j}} p_{ij}, \quad \mathbf{Q} = \mathbf{Q}^T.$$

- 可逆的含义:

固定n. $\{X_m : 0 \leqslant m \leqslant n\}$ 的(时间倒)逆过程为

$$\{Y_m : 0 \leqslant m \leqslant n\}, \quad \text{其中 } Y_m = X_{n-m}.$$



- $\{Y_m : 0 \leqslant m \leqslant n\}$ 的(有限维)轨道分布:

$$P(Y_0 = i_0, Y_1 = i_1, \ldots, Y_{n-1} = i_{n-1}, Y_n = i_n)$$
$$= P(X_0 = i_n, X_1 = i_{n-1}, \ldots, X_{n-1} = i_1, X_n = i_0)$$
$$= \pi_{i_n} p_{i_n i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0}.$$

- 逆过程$\{Y_m : 0 \leqslant m \leqslant n\}$ 的(有限维)轨道分布:

$$P(Y_0 = i_0, Y_1 = i_1, \ldots, Y_n = i_n) = \pi_{i_n} p_{i_n i_{n-1}} \ldots p_{i_1 i_0}.$$

- 如果π 是可逆分布, 即$\pi_i p_{ij} = \pi_j p_{ji}$, $\forall i, j$, 那么,

$$\begin{aligned}
&\pi_{i_n} p_{i_n i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} = \pi_{i_{n-1}} p_{i_{n-1} i_n} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} \\
=& p_{i_{n-1} i_n} \pi_{i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} \\
=& p_{i_{n-1} i_n} p_{i_{n-2} i_{n-1}} \pi_{i_{n-2}} \cdots p_{i_1 i_0} = \cdots \\
=& p_{i_{n-1} i_n} p_{i_{n-2} i_{n-1}} \cdots p_{i_0 i_1} \pi_{i_0} = \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}}.
\end{aligned}$$

- 如果初分布π 为可逆分布, 那么$\{X_n\}$ 与其逆过程同分布. 此时, 称$\{X_n\}$ 为可逆的.
- 可逆分布= 产生可逆过程.

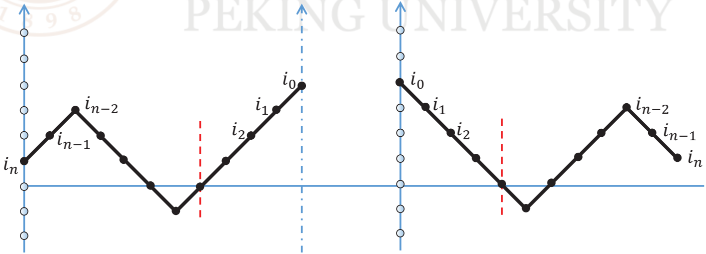
- 如果π 是不变分布, 那么$\hat{\mathbf{P}} = (\hat{p}_{ij})_{S\times S}$ 是转移矩阵,

$$\text{其中}, \ \hat{p}_{ij} := \frac{\pi_j p_{ji}}{\pi_i} \ \text{满足} \ \pi_j p_{ji} = \pi_i \hat{p}_{ij}, \ \forall i, j.$$

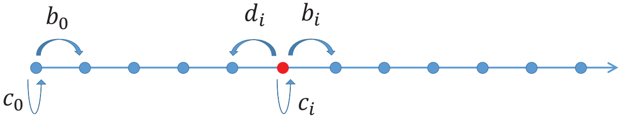
- 如果π 是不变分布, 那么,

$$\begin{aligned} \pi_{i_n} p_{i_n i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} &= \hat{p}_{i_{n-1} i_n} \pi_{i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} \\ &= \cdots = \pi_{i_0} \hat{p}_{i_0 i_1} \cdots \hat{p}_{i_{n-2} i_{n-1}}. \end{aligned}$$

- 马氏性: 已知现在(状态)的条件下, 将来与过去独立.



例1.2.10. 生灭链.



- $\pi_0 b_0 = \pi_1 d_1 \Rightarrow \pi_1 = \pi_0 \frac{b_0}{d_1}$.
- $\pi_1 b_1 = \pi_2 d_2 \Rightarrow \pi_2 = \pi_1 \frac{b_1}{d_2} = \pi_0 \frac{b_0 b_1}{d_1 d_2}$.
- $\pi_{i-1} b_{i-1} = \pi_i d_i \Rightarrow \pi_i = \pi_{i-1} \frac{b_{i-1}}{d_i} = \ldots = \pi_0 \frac{b_0 \ldots b_{i-1}}{d_1 \ldots d_i}$.
- 归一化：令

$$C := 1 + \frac{b_0}{d_1} + \frac{b_0 b_1}{d_1 d_2} + \ldots.$$

当$C < \infty$ 时，取

$$\pi_0 = \frac{1}{C}, \quad \pi_i = \frac{1}{C} \cdot \frac{b_0 \ldots b_{i-1}}{d_1 \ldots d_i}$$

即得可逆分布. 当$C = \infty$ 时，不变分布不存在.

求可逆分布的步骤:

(1) 固定o.

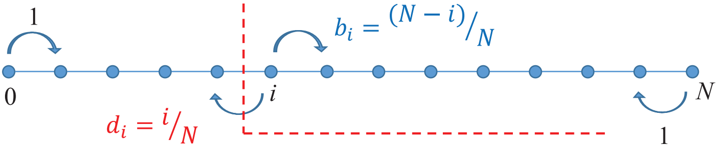
$\forall i \neq o$, 找$n \geqslant 1$ 以及$i_0 := o, i_1, \cdots, i_n := i$, 使得$p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0$, 并令

$$\pi_i = \pi_o \frac{p_{i_0 i_1} \cdots p_{i_{n-1} i_n}}{p_{i_1 i_0} \cdots p_{i_n i_{n-1}}}.$$

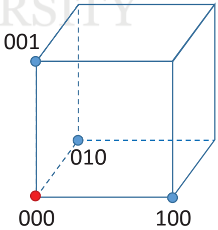
(2) 验证细致平衡条件: $\pi_i p_{ij} = \pi_j p_{ji}$, $\forall i, j$.

(3) 将π 归一化.

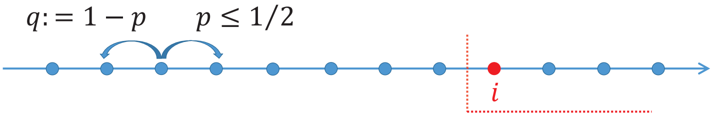
例1.1.8, 1.2.10 & 1.2.11 Ehrenfest模型.



- $\pi_i \times \frac{i}{N} = \pi_{i-1} \times \frac{N-(i-1)}{N}$.
- $\pi_i = \frac{N-i+1}{i}\pi_{i-1} = \ldots = \frac{(N-i+1)(N-i+2)\ldots(N+1-1)}{i(i-1)\ldots 1}\pi_0$.
  $= \frac{N!}{(N-i)!i!}\pi_0 = C_N^i \pi_0$.
- $\pi_0 \sum_{i=0}^N C_N^i = 1 \Rightarrow \pi_0 = \frac{1}{2^N}$.
  i.e. $\pi = B(N, \frac{1}{2})$.
- $Y_n \in \{0,1\}^N = \mathbb{H}_N$. $X_n = f(Y_n)$.
  $\mu_k = 2^{-N}, \forall k \in \mathbb{H}_N$.



例1.2.12. 一维紧邻随机游动. $p_{i,i+1} = 1 - p_{i,i-1} = p$.



- 可逆分布不存在. 否则,

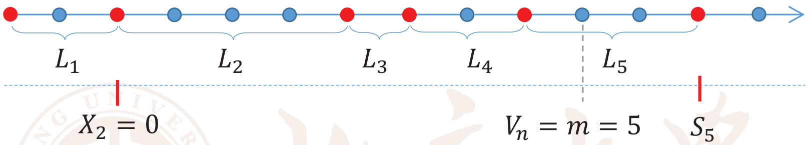
$$\pi_1 = \pi_0 \frac{p}{q}, \cdots \qquad \pi_{-1} = \pi_0 \frac{q}{p} = \pi_0 \frac{p^{-1}}{q^{-1}}, \cdots \qquad \pi_i = \pi_0 \frac{p^i}{q^i}, \ \forall i.$$

但, 上述π 不可以归一化.

- 不变分布不存在. 否则, 取$A = \{i, i+1 \ldots\}$, 则不变方程转变为细致平衡条件.
- $\mu_i = \mu_0, \forall i$ 是不变测度. 但, A 的总收$\neq$ 总支.
- $p = \frac{1}{2}$ 时, π 就是μ. 事实上, 不变测度 "唯一" .
- $p \neq \frac{1}{2}$ 时, π 不是μ. 不变测度 "不唯一" .

3. 访问频率

定理1.2.13 & 例1.2.14. 更新定理.



- 假设$EL < \infty$. 不变分布存在, 且$\pi_0 = \frac{1}{EL}$.
- $V_n = m$ iff $S_{m-1} \leqslant n < S_m$, 其中$S_m = L_1 + \ldots + L_m$.
- SLLN: $\frac{S_m}{m} \xrightarrow{\text{a.s.}} EL$

$$\Rightarrow \frac{n}{m} \xrightarrow{\text{a.s.}} EL$$

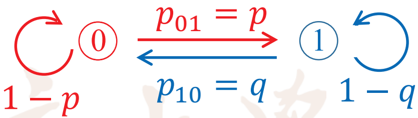
$$\Rightarrow \frac{V_n}{n} \xrightarrow{\text{a.s.}} \frac{1}{EL}, \text{(更新定理)}.$$

- $\{X_n\}$ 访问状态0 的频率 $\xrightarrow{\text{a.s.}} \pi_0$.

4. 访问概率的收敛性

例1.2.5. 两状态马氏链.

$$\mathbf{P}: \quad \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$



- $\pi_0 p = \pi_1 q \Rightarrow \pi_0 = \frac{q}{p+q}, \quad \pi_1 = \frac{p}{p+q}$.
- 特征向量: $\pi\mathbf{P} = \pi$, $\mathbf{P1} = \mathbf{1}$. (Perron-Frobenius 定理)
- $\mu \mapsto \mu\mathbf{P}$: $X_0 \sim \mu \Rightarrow X_1 \sim \mu P$.
- $f \mapsto \mathbf{P}f$: $(\mathbf{P}f)(i) = \sum_j p_{ij} f(j) = E_i f(X_1)$,

  $(\mathbf{P}^n f)(i) = E_i f(X_n)$.
- 不动点: 不变测度、调和函数.

- $\mathbf{P} = \begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \boxed{1-p-q} \end{pmatrix} \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{1}{p+q} & -\frac{1}{p+q} \end{pmatrix}$
- $\mathbf{P}^n \to \begin{pmatrix} 1 & \star \\ 1 & \star \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \boxed{0} \end{pmatrix} \begin{pmatrix} \pi_0 & \pi_1 \\ * & * \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}$
- $p_{ij}^{(n)} = P_i(X_n = j) \to \pi_j, \ \forall i, j.$
- $\mu \mathbf{P}^n \to (\mu_0, \mu_1) \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix} = (\pi_0, \pi_1).$
- $P(X_n = 0) \to \pi_0, \quad P(X_n = 1) \to \pi_1.$

- 不变分布π 的存在(唯一)性.
- 正常返(不可约).

- 对任意初始分布μ, 都有$\mu \mathbf{P}^n \to \pi$?
- 强遍历定理.

- 访问i 的频率$\to \pi_i$.
- 遍历定理.

§1.3 状态的分类

- 定义1.1.3. 可达, $i \to j$: $P_i(\exists n \geqslant 0$ 使得$X_n = j) > 0$.
- 或者$j = i$,

  或者$\exists n \geqslant 1, i_0, \cdots, i_n$,

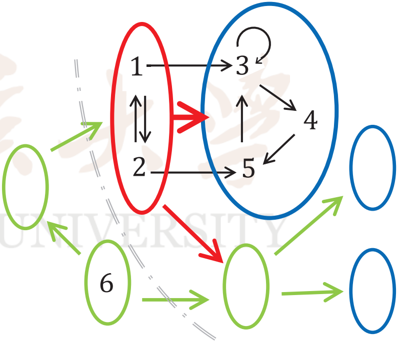
  $(i_0 = i, i_n = j)$,

  使得$p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0$.
- 存在$n \geqslant 0$ 使得$p_{ij}^{(n)} > 0$.

  证明: $P_i(X_n = j)$

  $\leqslant P_i(\exists m \geqslant 0$ 使得$X_m = j)$

  $\leqslant \sum_{m=0}^{\infty} P_i(X_m = j)$.



- 定义1.3.3. 互通$i \leftrightarrow j$: $i \to j$ 且$j \to i$.

  若所有i, j 都互通, 则称$\mathbf{P}$ (S, 马氏链)不可约, 否则称可约.

- 互通类: $[i] = \{j : j \leftrightarrow i\}$.

- 定义1.3.5. 若

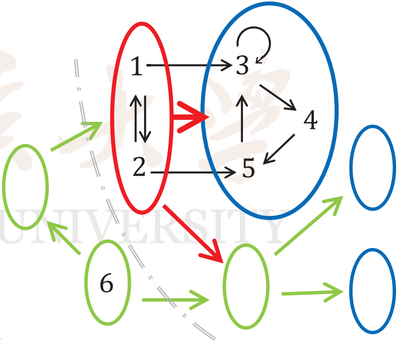
  $$p_{ij} = 0, \quad \forall i \in A, \ \forall j \notin A,$$

  则称A 为闭集/闭的.

  闭的互通类称为闭类.

- 命题1.3.6. 若A 为非空闭集, 则

  $$P_i(X_n \in A, \forall n \geqslant 0) = 1, \quad \forall i.$$



§1.4 强马氏性

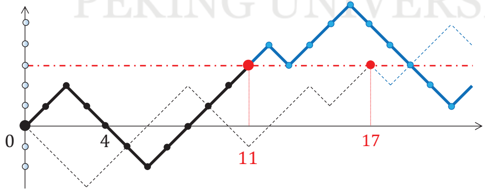
- 样本轨道: $\omega = \Big(X_0(\omega), X_1(\omega), X_2(\omega), \ldots\Big) = \vec{X}(\omega)$.
- 首达时τ_i 与首入时σ_i:

$$\tau_i(\omega) = \tau_i(\vec{X}) := \inf\{n \geqslant 0 : X_n = i\}.$$

$$\sigma_i(\omega) = \sigma_i(\vec{X}) := \inf\{n \geqslant 1 : X_n = i\}.$$

- τ_i, σ_i 是随机变量.

  例, $\tau_3(\omega) = 11$, $\tau_3(\hat{\omega}) = 17$; $\tau_0(\omega) = 0$, $\sigma_0(\omega) = 4$.

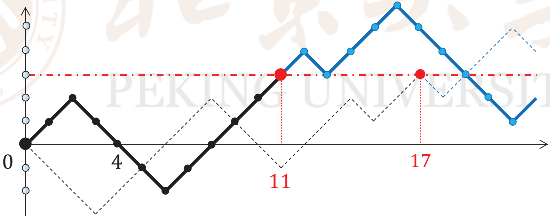


- $i \to j$: $P_i(\tau_j < \infty) > 0$.

- 固定i, 简记$\tau := \tau_i$. 在$\{\tau < \infty\}$ 上, 令

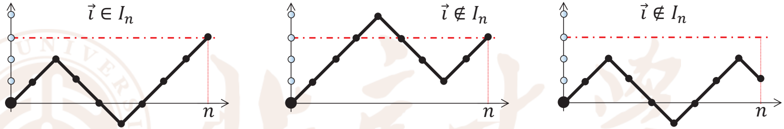
$$\vec{Y} = (Y_0, Y_1, Y_2, \ldots) := (X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots),$$

$$\vec{Z} := (X_0, \ldots, X_\tau).$$



- 固定n, 记$\vec{i} := (i_0, i_1, \ldots, i_n)$, 令

$$I_n := \left\{\vec{i} \in S^{n+1} : i_0, \ldots, i_{n-1} \neq i \text{ 且 } i_n = i\right\}.$$

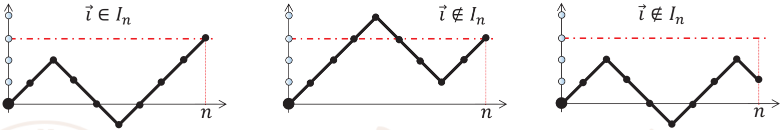


- 记$\vec{X}^{(n)} := (X_0, \ldots, X_n)$.
- 重要结论1: 若$\vec{i} \notin I_n$, 则

$$\{\vec{X}^{(n)} = \vec{i}\} \subseteq \{\tau \neq n\}, \quad \{\vec{Z} = \vec{i}\} = \varnothing.$$

- 在$\{\tau_i < \infty\}$ 上, $\vec{Z}$ 是离散型. 取值于$\bigcup_{n=0}^{\infty} I_n$.

- (i) 若$\vec{i} \in I_n$, 则$\vec{X}^{(n)} = \vec{i}$ 表明$\tau = n$, 从而$\vec{Z} = \vec{X}^{(n)} = \vec{i}$.



- (ii) $\vec{Z} = \vec{i}$ 表明$\tau = n$, 从而$\vec{X}^{(n)} = \vec{Z} = \vec{i}$.
- 重要结论2: 若$\vec{i} \in I_n$, 则

$$\{\vec{Z} = \vec{i}\} = \{\vec{X}^{(n)} = \vec{i}\} \subseteq \{\tau = n\}.$$

- 若$\vec{i} \in I_n$, 则

$$P(\vec{Z} = \vec{i}) = P(\vec{X}^{(n)} = \vec{i}) = \mu_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

命题 (命题1.4.2)

在$\{\tau_i < \infty\}$ 发生的条件下，

(1) $\{Y_n\}$ 是从i 出发的马氏链；(2) $\{Y_n\}$ 与$\vec{Z}$ 相互独立.

- 记$\tau_i = \tau$, $\vec{i} := (i_0, \ldots, i_n)$; $\vec{j} := (j_0, \ldots, j_m)$,

$$B := \{(Y_0, \ldots, Y_m) = \vec{j}\} = \{(X_\tau, \ldots, X_{\tau+m}) = \vec{j}\},$$
$$C := \{\vec{Z} = \vec{i}\} = \{(X_0, \ldots, X_\tau) = \vec{i}\}.$$

- 只需验证:

$\forall n \geqslant 0$, $\forall \vec{i} \in I_n$ (由重要结论1)； $\forall m \geqslant 0$, $\forall \vec{j} \in S^{m+1}$,

$$P_{\{\tau<\infty\}}(BC) = 1_{\{j_0=i\}} p_{j_0 j_1} \cdots p_{j_{m-1} j_m} \times P_{\{\tau<\infty\}}(C).$$

- $B=\{(X_\tau,\dots,X_{\tau+m})=\vec{j}\}$, $C=\{\vec{Z}=\vec{i}\}$.
- 回顾重要结论2: 若$\vec{i}\in I_n$, 则

$$\{\vec{Z}=\vec{i}\}=\{\vec{X}^{(n)}=\vec{i}\}\subseteq\{\tau=n\}.$$

- 令$\hat{B}:=\{(X_n,\dots,X_{n+m})=\vec{j}\}$, $\hat{C}:=\{\vec{X}^{(n)}=\vec{i}\}$.
- 由重要结论2知, 因为$\vec{i}\in I_n$, 所以$C=\hat{C}$. 从而,

$$BC=B\hat{C}\cap\{\tau=n\}=\hat{B}\hat{C}\cap\{\tau=n\}=\hat{B}\hat{C}.$$

- 于是

$$P_{\{\tau<\infty\}}(BC)=\frac{P(BC)}{P(\tau<\infty)}=\frac{P(\hat{B}\hat{C})}{P(\tau<\infty)}$$

$$=\frac{P(\hat{C})\times 1_{\{j_0=i\}}p_{j_0j_1}\cdots p_{j_{m-1}j_m}}{P(\tau<\infty)}=\star\star\times P_{\{\tau<\infty\}}(C).$$

- 重要结论:

  若$\vec{i} \in I_n$, 则

$$\{(X_0, \ldots, X_n) = \vec{i}\} \subseteq \{\tau = n\}. \tag{1}$$

若$\vec{i} \notin I_n$, 则

$$\{(X_0, \ldots, X_n) = \vec{i}\} \subseteq \{\tau \neq n\}. \tag{2}$$

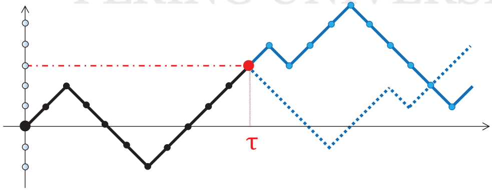
- 若存在$I_n \subseteq S^{n+1}$ 使得重要结论成立, 则称τ 为停时.

  即, $\forall \vec{i} \in S^{n+1}$, (1) 与(2) 之一成立, 则$\tau$ 为停时. (命题1.4.7)
- τ 为停时iff $\{\tau \leqslant n\} \in \sigma(X_0, \cdots, X_n) =: \mathcal{F}_n, \forall n$. (定义1.4.6)
- $\tau_i, \quad \tau_D := \inf_{i \in D} \tau_i; \qquad \sigma_i, \quad \sigma_D := \inf_{i \in D} \sigma_i$ 都是停时.
- $\varepsilon_i = \sup\{n \geqslant 0 : X_n = i\}$ 不是停时.

- 在$\{\tau < \infty\}$ 上, 令

$$\vec{Y} = (Y_0, Y_1, Y_2, \ldots) := (X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots),$$
$$\vec{Z} := (X_0, \ldots, X_\tau).$$

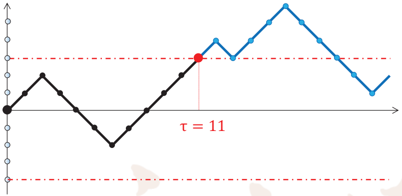
- 强马氏性(命题1.4.10).
假设τ 是停时. 那么, 在$\{\tau < \infty, X_\tau = i\}$ 发生的条件下, (1) $\{Y_n\}$ 是从i 出发的马氏链, (2) $\{Y_n\}$ 与$\vec{Z}$ 相互独立.
- 拼接轨道:



例. 一维简单随机游动(SRW) $\{S_n\}$.

- 固定$i, j \geqslant 1$, 记

  $\tau := \tau_i \wedge \tau_{-j}$.



- 以后会证明:

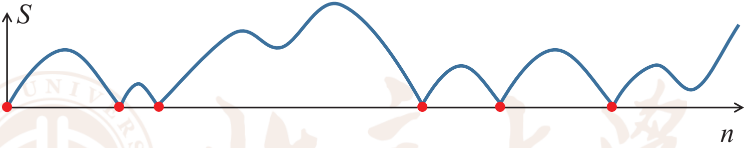
$$P(\tau < \infty) = 1, \quad P(S_\tau = i) = \frac{j}{i+j} =: p.$$

- $\{Y_n := S_{\tau+n}\}$ 也是SRW, 初分布为$\mu_i = p$, $\mu_{-j} = 1 - p =: q$.
- 对于轨道集A,

$$P(\vec{Y} \in A) = pP(\vec{Y} \in A | S_\tau = i) + qP(\vec{Y} \in A | S_\tau = -j)$$
$$= pP_i(A) + qP_{-j}(A).$$

§1.5 常返性

- 固定i. 轨道图:



- 令$V_i = \sum_{n=0}^{\infty} 1_{\{X_n=i\}}$.

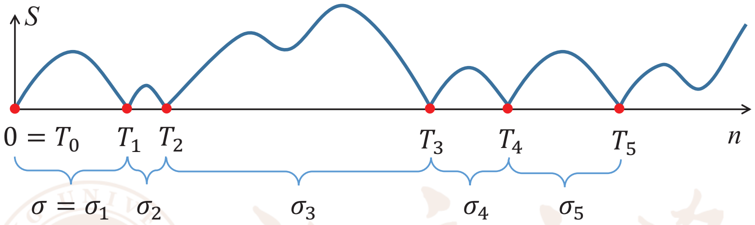
定义 (定义1.5.3))

若$P_i(V_i = \infty) = 1$, 则称i 是常返的(recurrent).

若$P_i(V_i = \infty) = 0$, 则称i 是非常返的(transient).

- 0-1律、等价性、判别法.

1. 回访时间



- 固定i. 令$T_0 := 0$. $\forall r \geqslant 1$,

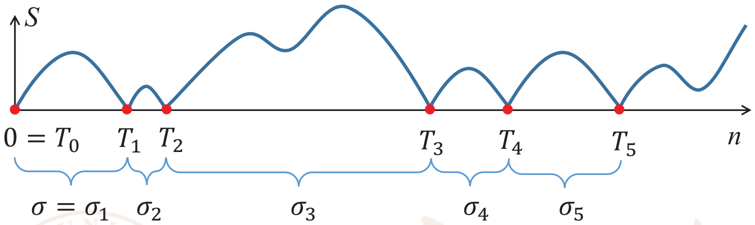
  若$T_{r-1} < \infty$, 则

  $$T_r := \inf\{n \geqslant T_{r-1} + 1 : X_n = i\}, \quad \sigma_r := T_r - T_{r-1};$$

  若$T_{r-1} = \infty$, 则$T_r := \infty$, $\sigma_r := \infty$.

- 常返$\Leftrightarrow$ $P_i(T_r < \infty, \forall r) = 1$.

  关键量: $\sigma = \sigma_1 := \inf\{n \geqslant 1 : X_n = i\}$.

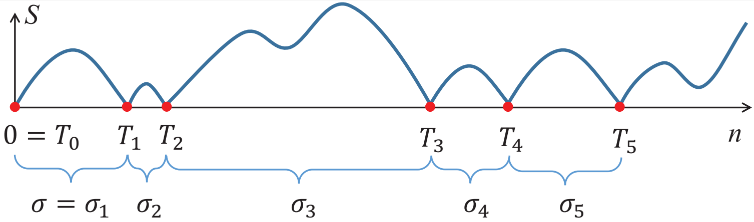


- 固定n, 记$\vec{i} := (i_0, i_1, \ldots, i_n)$, 令

$$J_n := \left\{ \vec{i} \in S^{n+1} : i_0 = i_n = i, \text{ 且 } i_r \neq i,\ r = 1, \cdots, n-1 \right\}.$$

- 在$\{\sigma < \infty\}$ 上, $\vec{Z}$ 是离散型, 取值于$\bigcup_{n=1}^{\infty} J_n$.
- $P(\vec{Z} = \vec{i}) = p_{i_0 i_1} \cdots p_{i_{n-1} i_n}, \quad \forall \vec{i} \in J_n$.

$$P(\vec{Z} \in J_n) = P_i(\sigma = n), \quad P(\vec{Z} \in J) = P_i(\sigma < \infty).$$



- 命题1.5.1. $\forall r \geqslant 1;\ m_1,\ldots,m_r \geqslant 1,$

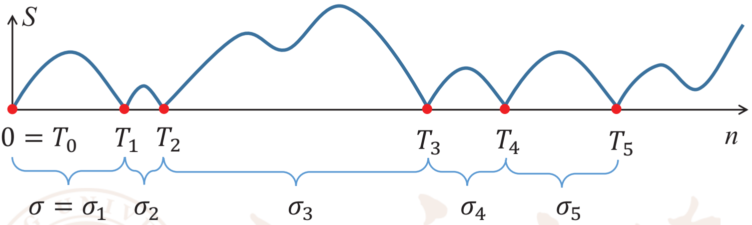
$$P_i(\sigma_1 = m_1, \sigma_2 = m_2, \cdots, \sigma_r = m_r) = \prod_{s=1}^{r} P_i(\sigma = m_s).$$

- 证: 令$n_s = \sum_{t=1}^{s} m_t;\ \vec{X}_s = (X_{n_{s-1}}, \cdots, X_{n_s})$. 则

$$\text{LHS} = \sum_{\vec{i}_1 \in J_{m_1}, \cdots, \vec{i}_r \in J_{m_r}} P_i\left(\vec{X}_1 = \vec{i}_1, \cdots, \vec{X}_r = \vec{i}_r\right)$$

$$= \prod_{s=1}^{r} \sum_{\vec{i}_s \in J_{m_s}} P_i\left(\vec{X}_s = \vec{i}_s\right) = \text{RHS}.$$

2. 回访次数



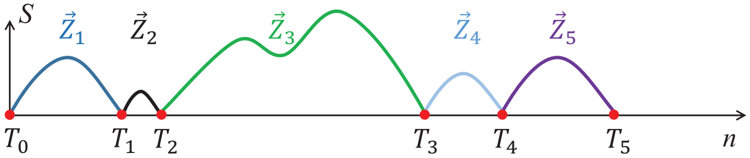
- 记$\rho_i = P_i(\sigma < \infty) = \sum_{m=1}^{\infty} P_i(\sigma = m)$.
- $P_i(V_i \geqslant r+1) = P_i(T_r < \infty) = \rho_i^r$ (推论1.5.2).

$$P_i(T_r < \infty) = P_i(\sigma_1 < \infty, \cdots, \sigma_r < \infty) = \rho_i^r.$$

- **0-1律**:

  若$\rho_i = 1$, 则$P_i(V_i = \infty) = 1$ (常返), $E_i V_i = \infty$.

  若$\rho_i < 1$, 则$P_i(V_i < \infty) = 1$ (非常返), $E_i V_i < \infty$.



- 假设i 常返, $X_0 = i$. 令

$$\vec{Z}_r = (X_{T_{r-1}}, \cdots, X_{T_r}).$$

- 命题1.5.4. $\vec{Z}_1, \vec{Z}_2, \cdots$ 独立同分布, 且$\vec{Z}_r$ 的分布列如下:

$$P(\vec{Z} = \vec{i}) = p_{i_0 i_1} \cdots p_{i_{n-1} i_n}, \quad \forall n \geqslant 1,\ \vec{i} = (i_0, \cdots, i_n) \in J_n.$$

- 证: $\forall r \geqslant 2$, $\forall \vec{i}_1, \cdots, \vec{i}_r \in \bigcup_{n=1}^{\infty} J_n$,

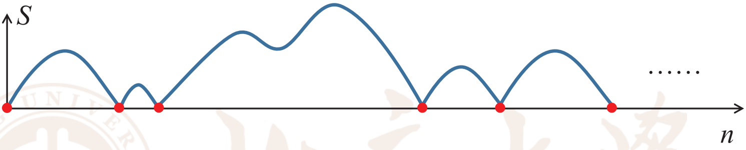
  考虑$\{\vec{Z}_1 = \vec{i}_1, \vec{Z}_2 = \vec{i}_2, \cdots, \vec{Z}_r = \vec{i}_r\}$. 由强马氏性,

$$P(B|C) = P(B|A, C) = P_i(\vec{Z}_1 = \vec{i}_2, \ldots, \vec{Z}_{r-1} = \vec{i}_r).$$

  其中, $A = \{\sigma = |\vec{i}_1|, X_\sigma = i\}$.

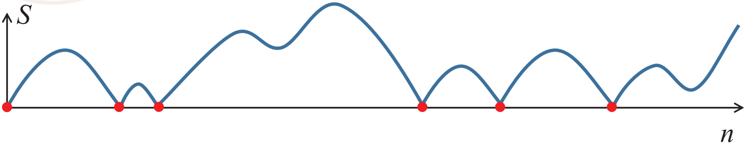
i 常返的等价条件:

- i 常返$\Leftrightarrow \rho_{ii} = 1 \Leftrightarrow E_iV_i = \infty$.



由i.i.d. 游弋(excursion)拼接.

- i 非常返$\Leftrightarrow \rho_{ii} < 1 \Leftrightarrow E_iV_i < \infty$.

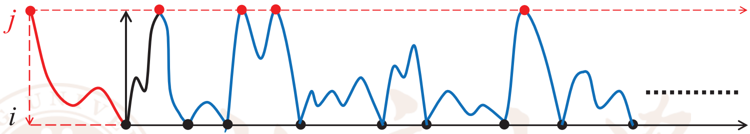


- 判别法: (1)定义、(2)吸收概率、(3)格林函数.

3. 常返是互通类的性质

命题1.5.5. 设i 常返且$i \to j$. 则, ······

- 设$X_0 = i$, 则$\vec{Z}_1, \vec{Z}_2, \ldots$ i.i.d..



- $P_i(\sigma_j < \infty) = P_i(V_j = \infty) = 1$:

$$P_i(\exists r \text{ 使得} j \in \vec{Z}_r) = P_i(\sigma_j < \infty) > 0 \Rightarrow P_i(j \in \vec{Z}_1) > 0.$$

- $P_j(V_i = \infty) = 1$: 进一步, 再由强马氏性,

$$P_j(V_i = \infty) = P_i(V_i = \infty, \sigma_j < \infty) = P_i(V_i = \infty) = 1$$

- $P_j(V_j = \infty) = 1$, 等价性:

$$P_j(V_j = \infty) = P_j(\sigma_i < \infty, V_j = \infty) = P_j(V_j = \infty) = 1.$$

4. 吸收概率与首步分析法

- 吸收态: $p_{ii} = 1$ (定义1.5.6).
- 吸收概率: 给定$o \in S$, 令

$$x_i = P_i(\tau_o < \infty), \quad \forall i \in S.$$

- 一般地, 给定$D \subseteq S$, 令

$$x_i = P_i(\tau_D < \infty), \quad \forall i \in S.$$

- x_i, $i \in S$ 是下列方程组的最小非负解,

$$\begin{cases} x_i = \sum_{j\in S} p_{ij} x_j, & \forall\ i \notin D, \\ x_i = 1, & \forall\ i \in D. \end{cases}$$

- 命题1.5.7. 若$\tilde{x}_i$, $i \in S$ 也是方程组的非负解, 则$\tilde{x}_i \geqslant x_i$, $\forall i$.

$$\begin{aligned}\tilde{x}_i =& \sum_{j\in D} p_{ij}\tilde{x}_j + \sum_{j\notin D} p_{ij}\tilde{x}_j \quad (i \notin D)\\ =& P_i(\tau_D = 1) + \underline{\sum_{j\notin D} p_{ij}}\Big(\underline{\sum_{k\in D} p_{jk}\tilde{x}_k} + \sum_{k\notin D} p_{jk}\tilde{x}_k\Big)\\ =& P_i(\tau_D = 1) + \underline{P_i(\tau_D = 2)} + \sum_{i_1,i_2\notin D} p_{ii_1}p_{i_1i_2}\tilde{x}_{i_2}\\ =& \cdots \geqslant P_i(\tau_D \leqslant r) \to x_i.\end{aligned}$$

- $\rho_o = P_o(\sigma_o < \infty) = \sum_{i\in S} p_{oi}P_i(\tau_o < \infty)$.

- 判别法: 假设不可约. 令$x_i = P_i(\tau_o < \infty)$, $i \in S$.

$$\text{常返 iff } x_i = 1, \ \forall i \in S.$$

5. 格林函数

- $V_i = \sum_{n=0}^{\infty} 1_{\{X_n=i\}}$.
- 格林函数:

$$G_{ij} := E_i V_j = E_i \sum_{n=0}^{\infty} 1_{\{X_n=j\}} = \sum_{n=0}^{\infty} p_{ij}^{(n)}.$$

- 常返 iff $G_{ii} = \infty$.

例1.5.11. $\mathbb{Z}^d$ 上的简单随机游动.

- $d = 1$. $p_{00}^{(2n+1)} = 0$,

$$p_{00}^{(2n)} = C_{2n}^n \frac{1}{2^{2n}} = \frac{(2n)!}{n!n!} \cdot \frac{1}{2^{2n}} \approx \frac{(\frac{2n}{e})^{2n}\sqrt{2\pi 2n}}{(\frac{n}{e})^{2n} 2\pi n} \cdot \frac{1}{2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

$G_{ii} = \sum_{n=0}^{\infty} p_{00}^{(2n)} = \infty$. 常返.

- $d = 2$.

$$p_{00}^{(2n)} = \sum_{n_1+n_2=n} \frac{(2n)!}{(n_1!)^2(n_2!)^2} \frac{1}{4^{2n}}$$

$$= C_{2n}^n \frac{1}{2^{2n}} \sum_{m=0}^{n} C_n^m C_n^{n-m} \frac{1}{2^{2n}} = (C_{2n}^n \frac{1}{2^{2n}})^2 \approx \frac{1}{\pi n}.$$

$G_{ii} = \sum_{n=0}^{\infty} p_{00}^{(2n)} = \infty$. 常返.

例1.5.11. $\mathbb{Z}^d$ 上的简单随机游动(续).

- $d \geqslant 3$. $p_{00}^{(2n)} = \sum_{n_1+\ldots+n_d=n} \frac{(2n)!}{(n_1!)^2 \ldots (n_d!)^2} \times \frac{1}{(2d)^{2n}}$.
- $P(\xi_1 = 2n_1, \ldots, \xi_d = 2n_d) = \frac{(2n)!}{(2n_1)! \ldots (2n_d)!} \frac{1}{d^{2n}}$.

  $P(B_i) = C_{2n_i}^{n_i} \frac{1}{2^{2n_i}}, \ i = 1, \cdots, d.$
- $p_{00}^{(2n)} = \sum_{n_1+\ldots+n_d=n} P(A) P(B_1) \ldots P(B_d)$.
- SLLN $\Rightarrow n_1, \ldots, n_d \approx \frac{n}{d}$. 具体地,

$$\vec{n} \in I: \ |n_i - \tfrac{n}{d}| \leqslant \varepsilon n, \ i = 1, \cdots, d; \quad J = \mathbb{Z}_+^d \backslash I.$$

- 若$\vec{n} \in I$, 则

$$P(B_i) \approx \frac{1}{\sqrt{\pi n_i}} \approx \frac{1}{\sqrt{\pi n / d}}.$$

- $\sum_J P(A) \times 1 \leqslant d \cdot P(|\xi_1 - \frac{n}{d}| > \varepsilon n) \leqslant e^{-\bar{C} n}$.
- $p_{00}^{(2n)} \leqslant \sum_I \hat{C}^d / \sqrt{n}^d + \sum_J P(A) \leqslant C n^{-d/2}, \ \ n \gg 1$. 非常返.

例. 区域D中的格林函数. 固定$D \subseteq S$, 令$\tau = \tau_{D^c}$,

$$G_{ij}^{(D)} = E_i \sum_{n=0}^{\tau-1} 1_{\{X_n=j\}}, \qquad i,j \in D.$$

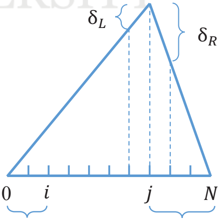
- 固定j. $x_i = G_{ij}^{(D)}$, $i \in D$ 是下列方程组的最小非负解,

$$\begin{cases} x_i = \sum_k p_{ik} x_k, & \forall\, i \in D,\ i \neq j, \\ x_j = 1 + \sum_k p_{jk} x_k, & \\ x_i = 0, & \forall\, i \notin D. \end{cases}$$

- 特例. 一维简单随机游动. $D = (0, N)$.

$\delta_L = \frac{1}{j} x_j; \qquad \delta_R = \frac{1}{N-j} x_j.$

$0 = -1 + \delta_L/2 + \delta_R/2.$



- $G_{ij}^{(N)} = G_{ji}^{(N)} = \frac{2}{N}(i \wedge j)(N - i \vee j).$

例. 区域D 中的格林函数(续). $i,j\in D$.

- $G_{ij}^{(D)}=E_i\sum_{n=0}^{\tau-1}1_{\{X_n=j\}}\neq\sum_{n=0}^{\tau-1}P_i(X_n=j)$.
- $G_{ij}^{(D)}=E_i\sum_{n=0}^{\infty}1_{\{X_n=j,\tau>n\}}=\sum_{n=0}^{\infty}P_i(X_n=j,\tau>n)$.
- 令$I_n:=\{\vec{i}=(i_0,\ldots,i_n)\in S^{n+1}:i_0,\ldots,i_n\in D\}$.

$$\star\star=\sum_{\vec{i}\in I_n:i_0=i,i_n=j}p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n}.$$

- $\mathbf{G}^{(D)}=\sum_{n=0}^{\infty}\hat{\mathbf{P}}^n=(\mathbf{I}-\hat{\mathbf{P}})^{-1}$. 其中$\hat{\mathbf{P}}=\mathbf{P}|_{D\times D}$.
- 若有配称测度π, 则$\pi_iG_{ij}^{(D)}=\pi_jG_{ji}^{(D)}$.

- $G_{ii} = \sum_{n=0}^{\infty} p_{ii}^{(n)}$.

$$P_i(X_n = i) = \sum_{m=1}^{n} P_i(\sigma_i = m) P_i(X_{n-m} = i), \quad n \geqslant 1.$$

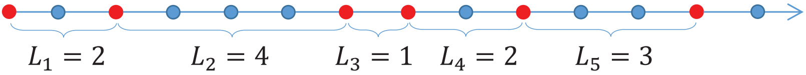
- 令$G_{ii}(s) = \sum_{n=0}^{\infty} p_{ii}^{(n)} s^n$. 那么,

$$G_{ii}(s) - 1 = \sum_{n=1}^{\infty} \sum_{m=1}^{n} P_i(\sigma_i = m) s^m p_{ii}^{(n-m)} s^{n-m}$$

$$= \sum_{m=1}^{\infty} P_i(\sigma_i = m) s^m \sum_{\ell=0}^{\infty} p_{ii}^{(\ell)} s^{\ell} = F_{ii}(s) G_{ii}(s).$$

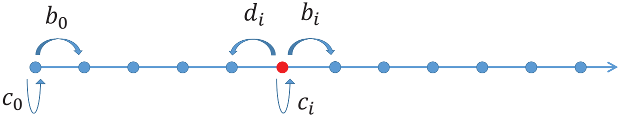
- 例1.1.9. $p_m = P(L = m)$, $a_n = P_0(Y_n = 0)$. 更新方程:

$$a_n = \sum_{m=1}^{n} p_m a_{n-m}.$$



- 可配称马氏链 $\Leftrightarrow$ 加权图(上的随机游动) $\Leftrightarrow$ 电网络:
  $G=(V,E)$: $V=S$, $i\overset{e}{\sim}j \Leftrightarrow p_{ij}>0$.
  电阻、权:
  $$r_{ij}=r_{ji}=\frac{1}{w_{ij}}=\frac{1}{\pi_i p_{ij}},\quad i\sim j.$$
- 取定$o\in S$. 吸收概率法: $x_i=P_i(\tau_o<\infty)$ 满足:
  $$x_o=1;\ \ x_i=\sum_{j\in S}p_{ij}x_j,\ \forall\ i\neq o.$$
  $$\sum_j \pi_i p_{ij}(x_j-x_i)=0.$$
- 在o 接电势为1 的电源: 电势V_i 满足:
  $$V_o=1;\ \ \sum_{j\in S}\frac{1}{r_{ij}}(V_j-V_i)=0,\ \forall\ i\neq o.$$
- 命题1.6.2. 常返$\Leftrightarrow$ $R=R(0,\infty)=\infty$.
  $$R=I^2R=\inf\{\textstyle\sum_e f_e^2 r_e : f \text{ 是从}o\text{ 出发的单位流}\}.$$

例1.6.5. 生灭链.



- 可配称. 并且

$$\frac{r_{i,i+1}}{r_{i,i-1}} = \frac{w_{i,i-1}}{w_{i,i+1}} = \frac{d_i}{b_i}$$

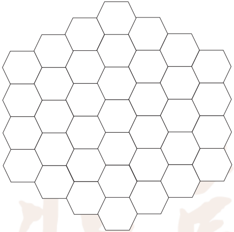
$$\Rightarrow r_{i,i+1} = \frac{d_i}{b_i} r_{i-1,i} = \cdots = \frac{d_i}{b_i} \cdots \frac{d_1}{b_1} r_{0,1}.$$

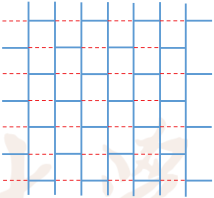
- 串联. 因此, 常返iff $\sum_i \frac{d_1}{b_1} \cdots \frac{d_i}{b_i} = \infty$.

推论1.6.4. $G_1 \subseteq G_2$. 若G_2 常返, 则G_1 常返.

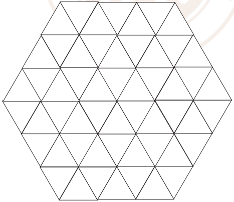
例: 已知$\mathbb{Z}^2$ 常返.

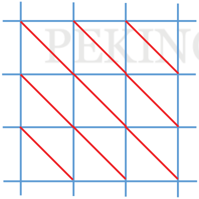
- 蜂窝图常返

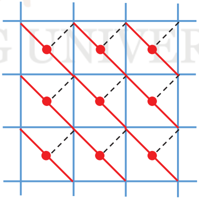


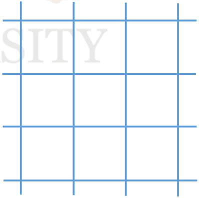


- 三角图常返









例1.6.9. $\mathbb{Z}^d$ 非常返.

- $d=3$ 时, 找单位流f, 使得$\sum_e f_e^2 r_e < \infty$. 则非常返.
- Polya 坛子: (X_n, Y_n, Z_n), $X_0 = Y_0 = Z_0 = 1$. 例如:

$$\vec{i} = (i,j,k),\ \vec{j} = (i+1,j,k), \quad p_{\vec{i},\vec{j}} = \frac{i}{i+j+k}$$

- $(X_n, Y_n, Z_n) \sim U(S_n)$,

$$S_n = \{\vec{i} = (i,j,k) : i,j,k \geqslant 1 \text{ 且} i+j+k = 3+n\}.$$

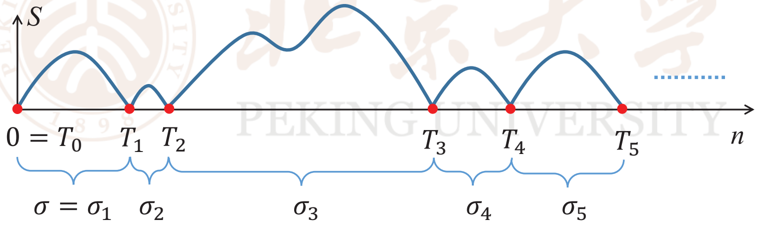
- 概率转移流: $f_{\vec{i},\vec{j}} = \pi_{\vec{i}} \cdot p_{\vec{i},\vec{j}}$.
- $\sum_e f_e^2 r_e < \infty$:

$$\sum_{\vec{i} \in S_n} \frac{1}{|S_n|^2} \cdot \frac{i^2+j^2+k^2}{(i+j+k)^2} \leqslant \frac{1}{|S_n|} = O\left(\frac{1}{n^2}\right).$$

1. 频率的极限

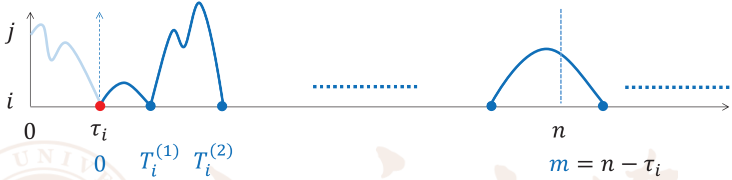
定理 (定理1.7.1)

假设**P** 不可约. 则 $P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}1_{\{X_m=i\}}=\frac{1}{E_i\sigma_i}\right)=1.$



- 假设非常返. 那么, 左右都为0.
- 假设常返. 假设$X_0=i$, 用更新定理即可.

- 假设 $X_0 = j$: $P_j(\tau_i < \infty) = 1$.



- $\omega = (\tilde{\omega}, \hat{\omega})$. $\quad n = m + \tau_i(\tilde{\omega})$, $\quad V_i(n, \omega) = V_i(m, \hat{\omega})$.
- $n \to \infty$ iff $m \to \infty$:

$$\frac{1}{n} V_i(n, \omega) = \frac{m}{n} \times \frac{1}{m} V_i(m, \hat{\omega}) \to \frac{1}{E_i \sigma_i}.$$

- 假设 $X_0 \sim \mu$:

$$P\left(\lim_{n \to \infty} \frac{1}{n} V_i(n) = \frac{1}{E_i \sigma_i} \right) = \sum_j \mu_j P_j(A) = 1.$$

命题 (命题1.7.2, 频率→ 概率)

假设**P** 不可约, π 为不变分布. 则

$$\pi_i = \frac{1}{E_i\sigma_i} > 0, \quad \forall i \in S.$$

- $\pi_i > 0$: $\pi_i = \sum_j \pi_j p_{ji}^{(n)} \geqslant \pi_o p_{oi}^{(n)} > 0.$
- 由遍历定理,

$$P_\pi\left(\lim_{n\to\infty} \frac{1}{n}V_i(n) = \frac{1}{E_i\sigma_i}\right) = 1.$$

- 再由有界收敛定理,

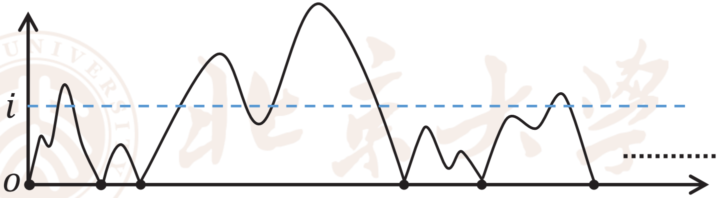
$$\lim_{n\to\infty} E_\pi \frac{1}{n}V_i(n) = \frac{1}{E_i\sigma_i}.$$

- π 是不变分布:

$$E_\pi \frac{1}{n}V_i(n) = E_\pi \frac{1}{n}\sum_{m=0}^{n-1} 1_{\{X_m=i\}} = \frac{1}{n}\sum_{m=0}^{n-1} P_\pi(X_m = i) = \pi_i.$$

2. 正常返与不变分布

- 再看频率: 固定o. 记$\sigma=\sigma_o$.



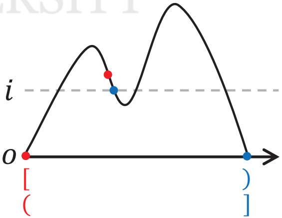
$$i\text{出现的频率}\approx\frac{V_i^{(1)}+\ldots+V_i^{(r)}}{\sigma_1+\ldots+\sigma_r}\approx\frac{E_oV_i^{(1)}}{E_o\sigma}\propto E_oV_i^{(1)}.$$

命题 (命题1.7.4 & 1.7.20)

假设不可约、常返. 固定o, 令$\mu_i = E_o \sum_{n=0}^{\sigma_o - 1} 1_{\{X_n = i\}}$. 那么, μ 是唯一$^{(c)}$满足$\mu_o = 1$ 的不变$^{(a)}$测度$^{(b)}$.

证明, (a) 不变性, 即, 验证$\sum_j \mu_j p_{ji} = \mu_i$. (记$\sigma = \sigma_o$.)

- $\mu_j = E_o \sum_{n=0}^{\infty} 1_{\{n<\sigma, X_n = j\}} = \sum_{n=0}^{\infty} P_o(n < \sigma, X_n = j)$.
- 于是, $\sum_j \mu_j p_{ji} = \sum_j \sum_{n=0}^{\infty} P_o(n < \sigma, X_n = j)$
  $\times P(X_{n+1} = i | X_n = j, n < \sigma)$
- $= \sum_{n=0}^{\infty} \sum_j P_o(n < \sigma, X_n = j, X_{n+1} = i)$
  $= \sum_{n=0}^{\infty} P_o(n < \sigma, X_{n+1} = i)$.
- $= E_o \sum_{n=0}^{\sigma-1} 1_{\{X_{n+1} = i\}} = E_o \sum_{n=1}^{\sigma} 1_{\{X_n = i\}}$
- $\overset{(*)}{=} E_o \sum_{n=0}^{\sigma-1} 1_{\{X_n = i\}} = \mu_i$.



证明(续), (b) 略. (c) 唯一性: 假设$\lambda\mathbf{P}=\lambda$ 且$\lambda_o=1$.

- 方法二、令$\hat{p}_{ij}=\frac{\mu_j p_{ji}}{\mu_i}$. 则$\hat{p}_{ij}^{(n)}=\frac{\mu_j p_{ji}^{(n)}}{\mu_i}$.
- $\hat{p}_{ii}^{(n)}=p_{ii}^{(n)}$, 所以$\{\hat{X}_n\}$ 也常返.
- 令$f(i)=\frac{\lambda_i}{\mu_i}$, $\forall i$. 则

$$(\hat{\mathbf{P}}f)(i)=\sum_j \hat{p}_{ij}f(j)=\sum_j \frac{\mu_j p_{ji}}{\mu_i}\cdot\frac{\lambda_j}{\mu_j}=\sum_j \frac{\lambda_j p_{ji}}{\mu_i}=f(i).$$

- $M_n=f(\hat{X}_n)$, $n\geqslant 0$ 是鞅:

$$E(M_{n+1}|\hat{X}_n=i,\hat{X}_0=i_0,\ldots,\hat{X}_{n-1}=i_{n-1})$$
$$=\sum_j \hat{p}_{ij}f(j)=(\hat{\mathbf{P}}f)(i)=f(i)=M_n.$$

- 定理: 非负鞅a.s.收敛. 因此, $f(\hat{X}_n)$ a.s.收敛.
- $\{\hat{X}_n\}$ 常返$\Rightarrow f\equiv f(o)=1$.

定义 (定义1.7.3))

若$E_i\sigma_i<\infty$, 则称i 是正常返的.

若$P_i(\sigma_i<\infty)=1$ 但$E_i\sigma_i=\infty$, 则称i 是零常返的.

定理 (定理1.7.5)

假设不可约. 则下面几条等价: *(1)* 所有状态正常返, *(2)* 存在正常返态, *(3)* 存在不变分布. 此时, $\pi_i=\frac{1}{E_i\sigma_i}$, $\forall i\in S$.

- 推论1.7.7. 假设S 有限、不可约. 那么, 不变分布存在.

$$1=\sum_i\frac{1}{n}\sum_{m=0}^{n-1}1_{\{X_m=i\}}\to\sum_i\frac{1}{E_i\sigma_i}.$$

- 推论1.7.21. $\exists$ 不能归一化的不变测度, 则不变分布不存在.
- 频率:

$$i\text{出现的频率}\approx\frac{V_i^{(1)}+\ldots+V_i^{(r)}}{\sigma_1+\ldots+\sigma_r}\approx\frac{E_oV_i^{(1)}}{E_o\sigma}=\frac{1}{E_i\sigma_i}=\pi_i.$$

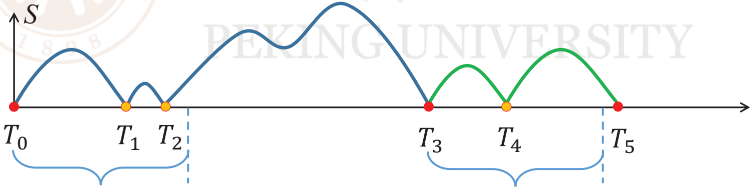
例1.7.17. 设不可约、正常返. 令$\sigma := \inf\{n \geqslant m : X_n = i\}$. 证明:

$$E_i \sum_{n=0}^{\sigma-1} 1_{\{X_n=j\}} = \pi_j E_i \sigma.$$

- 方法一、仿照命题1.7.5., 证明$\tilde{\mu}_i$, $i \in S$ 是不变测度.

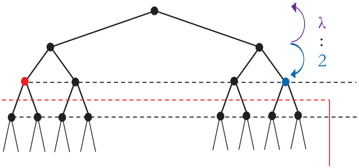
$$\tilde{\mu}_i = c\pi_i \Rightarrow c = \sum_i \tilde{\mu}_i = E_i \sigma.$$

- 方法二、令$S_0 = 0$, $S_r = \inf\{n \geqslant m + S_{r-1} : X_n = i\}$, 则



$$\text{频率} = \frac{\tilde{V}_j^{(1)} + \cdots + \tilde{V}_j^{(r)}}{\sigma_1 + \cdots + \sigma_r} \to \frac{\star\star}{E_i \sigma}.$$

例1.7.15. $\mathcal{T}^d$ 上λ-biased 随机游动:



- 对称性$\Rightarrow \pi_i = c_{|i|}$.
- $d^n c_n \times \frac{\lambda}{\lambda+d} = d^{n-1} c_{n-1} \times \frac{d}{\lambda+d}$,

$$c_n = \frac{1}{\lambda} c_{n-1} = \cdots \frac{1}{\lambda^n} c_0.$$

- 第n 层的总权重:

$$\sum_{|i|=n} \pi_i = \left(\frac{d}{\lambda}\right)^n c_0.$$

- 正常返$\Leftrightarrow \lambda > d$.
- 零常返$\Leftrightarrow \lambda = d$, 非常返$\Leftrightarrow \lambda < d$.

3. 遍历定理(假设不可约、正常返).

- 频率→ 概率:

$$\frac{1}{n}\sum_{m=0}^{n-1} 1_{\{X_m=i\}} \xrightarrow{\text{a.s.}} \pi_i = \frac{1}{E_i\sigma_i} = \frac{E_o V_i^{(1)}}{E_o\sigma_o}.$$

- 再由有界收敛定理:

$$\frac{1}{n}\sum_{m=0}^{n-1} P(X_m = i) \to \pi_i.$$

- 引理1.7.8: 如果$\nu^{(n)}$, $n \geqslant 1$ 都是分布列, $\nu_i^{(n)} \to \pi_i$, $\forall i$. 那么,

$$\sum_i |\nu_i^{(n)} - \pi_i| \to 0.$$

- 命题1.7.9.

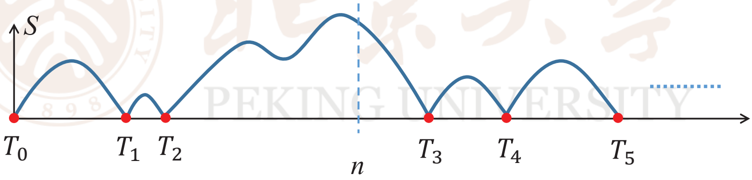
$$\sum_{i\in S}\left|\frac{1}{n}\sum_{m=0}^{n-1} p_{ji}^{(m)} - \pi_i\right| \to 0, \quad \sum_{i\in S}\left|\frac{V_i(n)}{n} - \pi_i\right| \xrightarrow{\text{a.s.}} 0.$$

定理 (遍历定理, 定理1.7.11)

假设不可约、正常返. 若$\sum_i \pi_i |f(i)| < \infty$, 则

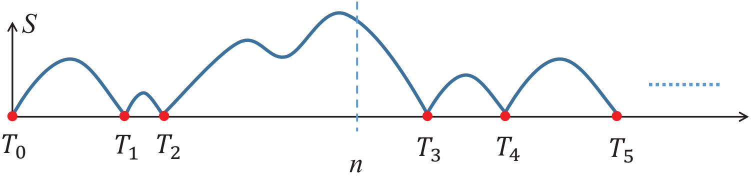
$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \xrightarrow{a.s.} \sum_i \pi_i f(i).$$

- 不妨设$X_0 = o$. 固定 o. $\xi_r = \sum_{n=T_{r-1}}^{T_r-1} f(X_n)$, $r \geqslant 1$ i.i.d..



- $T_r < n \leqslant T_{r+1}$:

$$\star\star = \xi_1 + \cdots + \xi_r + \sum_{m=T_r}^{n-1} f(X_m).$$



- $\xi=\sum_i f(i)V_i^{(1)}, \qquad \eta=\sum_i |f(i)|V_i^{(1)}.$

$$|\xi|\leqslant\eta, \qquad \max_{0\leqslant m\leqslant\sigma-1}\left|\sum_{n=0}^{m} f(X_n)\right|\leqslant\eta.$$

- 期望存在:

$$E\eta=\sum_i |f(i)|\underset{\sim\sim\sim\sim}{E_oV_i^{(1)}}=\sum_i |f(i)|\underset{\sim\sim\sim\sim}{\pi_i E_o\sigma}<\infty,$$

- $\frac{1}{n}(\xi_1+\cdots+\xi_r+\eta_r)\xrightarrow{\text{a.s.}}\sum_i f(i)\pi_i$:

$$\frac{\xi_1+\cdots+\xi_r}{r}\xrightarrow{\text{a.s.}}E\xi=\sum_i f(i)\pi_i E_o\sigma, \qquad \frac{\eta_r}{r}\xrightarrow{\text{a.s.}}0, \qquad \frac{r}{n}\to\frac{1}{E_o\sigma}.$$

补充：遍历.

- 保测变换: $\theta : (\Omega, \mathcal{F}, P) \hookleftarrow$, $P(\theta^{-1}(D)) = P(D)$.
  - 例：平稳马氏链, $\theta : \omega = (\omega_0, \omega_1, \omega_2, \ldots) \mapsto (\omega_1, \omega_2, \ldots)$.
    $\{(X_1, X_2, \ldots) \in D\} \overset{\text{w.p.}}{=} \{(X_0, X_1, \ldots) \in D\} = D$.
- 遍历：若$D \in \mathcal{F}$ 满足$D \subseteq \theta^{(-1)} D$, 则$P(D) = 0$ 或1.
  - 例：$\vec{X} \in D \Rightarrow \theta\vec{X} = (X_1, X_2, \ldots) \in D$.
  - 反例：$S = S_1 \cup S_2$, $\mu = \frac{1}{2}\pi + \frac{1}{2}\nu$, $D = \{X_n \in S_1, \forall n\}$.
  - 遍历$\Rightarrow$ 若$P(D) > 0$, 则$P(\bigcup_{n=0}^{\infty} \theta^{(-n)} D) = 1$.
- 遍历定理：遍历, $\xi : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, $E|\xi| < \infty$. 则

$$\frac{1}{n}\sum_{m=0}^{n-1} \xi\left(\theta^{(m)}(\omega)\right) \xrightarrow{\text{a.s.}} E\xi.$$

  - 例：不可约、正常返. 令$\xi(\omega) = f(X_0)$, 则$\star\star = f(X_m)$.

例1.7.16. 假设**P** 不可约、正常返. 则

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}1_{\{X_m=i,X_{m+1}=j\}}\xrightarrow{\text{a.s.}}\pi_i p_{ij}.$$

- 方法一、$\{Y_n=(X_n,X_{n+1})\}$ 是马氏链,
  转移概率: $p_{(i,j)(j,k)}=p_{jk}$, 不变分布: $\mu_{(i,j)}=\pi_i p_{ij}$. 故,

$$\frac{1}{n}\sum_{m=0}^{n-1}1_{\{X_m=i,X_{m+1}=j\}}=\frac{1}{n}\sum_{m=0}^{n-1}1_{\{Y_m=(i,j)\}}\xrightarrow{\text{a.s.}}\mu_{(i,j)}=\pi_i p_{ij}.$$

- 方法二: 假设n 之前完成r 次游弋,

$$\frac{r}{n}\times\frac{1}{r}\sum_{s=0}^{r-1}1_{\{X_{T_s+1}=j\}}\xrightarrow{\text{a.s.}}\pi_i p_{ij}.$$

§1.8 强遍历定理

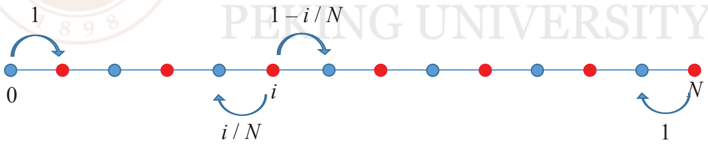
假设: 不可约、正常返.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}P(X_m=i)=\pi_i.$$

- $P(X_n=i)\approx\pi_i,\ \forall i?$ $(n\gg 1)$
- 反例: 例1.6.1. Ehrenfest模型. $S=\{0,1,\ldots,N\}$.

  $\pi_i=C_N^i 2^{-N},\ \forall i.$

  取$X_0=0$. $P(X_{2n}=1)=0,\ \forall n$.



- 希望: $p_{ij}^{(n)}>0,\ \forall n\gg 1$.

- 定义1.8.3: 若$\exists n_i$ 使得$p_{ii}^{(n)} > 0, \forall n \geqslant n_i$, 则称$i$ 为非周期的.
- 等价性: 如果$i \leftrightarrow j$ 且i 非周期, 则j 也非周期.
- 证: 存在r, s 使得$p_{ij}^{(r)}, p_{ji}^{(s)} > 0$, 于是

$$p_{jj}^{(r+s+n)} \geqslant p_{ji}^{(s)} p_{ii}^{(n)} p_{ij}^{(r)} > 0, \quad \forall n \geqslant n_i.$$

- 推论1.8.4. 若不可约、非周期, 则$\forall i, j$, $\exists n_{ij}$ 使得

$$p_{ij}^{(n)} > 0, \quad \forall n \geqslant n_{ij}.$$

- 假设不可约. 若存在i 使得$p_{ii} > 0$, 则非周期.

定理 (定理1.8.5. 强遍历定理)

假设不可约、正常返、非周期, 不变分布为π. 那么, $\forall \mu$,

$$\lim_{n\to\infty}\sum_i |P_\mu(X_n=i)-\pi_i|=0.$$

- 证明方法: 耦合(coupling).
- $\{Z_n=(W_n,Y_n)\}$: 状态空间、转移概率、初分布:

$$\tilde{S}=S\times S,\quad r_{(i,j)(k,\ell)}=p_{ik}p_{j\ell}.$$

- Step 1 ~ 3. $\{Z_n\}$ 不可约、非周期、(正)常返:

$$r^{(n)}_{(i,j)(k,\ell)}=p^{(n)}_{ik}p^{(n)}_{j\ell}>0,\quad \forall n\geqslant n_{ik}\vee n_{j\ell};$$

$$\tilde{\pi}=\pi\times\pi,\quad \text{i.e. } \tilde{\pi}_{i,j}=\pi_i\pi_j,\ \forall (i,j)\in\tilde{S}.$$

- Step 4. $P(\tau<\infty)=1$:

$$\tau=\inf\{n\geqslant 0: W_n=Y_n\}=\tau_D.$$

- Step 5. $P(W_n = j, \tau \leqslant n) = P(Y_n = j, \tau \leqslant n)$:

  $\forall 0 \leqslant m \leqslant n$, $i \in S$:

$$
\begin{aligned}
&P(W_n = j, \tau = m, Z_m = (i,i)) \\
=&P(*, *)P_{(i,i)}(W_n = j) = P(*, *)P_{(i,i)}(Y_n = j) \\
=&P(Y_n = j, \tau = m, Z_m = (i,i)).
\end{aligned}
$$

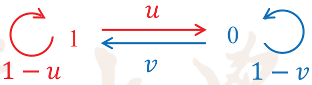
- Step 6. $P_i(X_n = j) - \pi_j = P_{\mu\times\pi}(W_n = j) - P_{\mu\times\pi}(Y_n = j)$:

$$
\begin{aligned}
|\star| =&|P(W_n = j, \tau > n) - P(Y_n = j, \tau > n)| \\
\leqslant&2P(\tau > n) \to 0.
\end{aligned}
$$

例1.8.6. Wright-Fisher模型. N 个人, 表态: “1” 或 “0” 每人每天独立任选一人并随机跟随或改变. $X_n=$ 第n 天表态1的人数. 求: $EX_n=\sum_{j=0}^N jP(X_n=j)$ 在$n\to\infty$ 时的极限.

- 若$X_n=i$, 则$X_{n+1}\sim B(N,\rho_i)$,

  其中, $\rho_i=(1-u)\frac{i}{N}+v\frac{N-i}{N}$.



- 不可约、正常返、非周期, $EX_n\to\sum_{j=0}^N j\pi_j$.
- $A=\sum_j j\sum_i \pi_i p_{ij}=\sum_i \pi_i \sum_j jp_{ij}=\sum_i \pi_i N\rho_i$

  $=\sum_i \pi_i[(1-u)i+v(N-i)]=(1-u-v)A+vN.$
- $A=\frac{v}{u+v}N.$
- $u=v=\frac{1}{2}$. $\rho_i=\frac{1}{2}$. $X_1,X_2,\ldots$ i.i.d., $\pi=B(N,1/2)$.

命题 (命题1.8.9)

若不可约、零常返、非周期，则$p_{ij}^{(n)} \to 0,\ \forall i,j.$

- Claim 1. $\forall \varepsilon > 0$, $\exists M$ 使得$\forall n$, 在

$$p_{ij}^{(n)}, p_{ij}^{(n+1)}, \cdots, p_{ij}^{(n+M)}$$

  中存在某个$p_{ij}^{(n+m)}$ 使得$p_{ij}^{(n+m)} < \varepsilon$.

- Claim 2. 当$n \gg 1$ 时, $|p_{ij}^{(n)} - p_{ij}^{(n+m)}| < \varepsilon,\ \forall 1 \leqslant m \leqslant M$.

- Claims $\Rightarrow p_{ij}^{(n)} < 2\varepsilon,\ n \gg 1$.

证明Claim 1. $\exists M$ 使得$\min\{p_{ij}^{(n)}, p_{ij}^{(n+1)}, \dots, p_{ij}^{(n+M)}\} < \varepsilon$, $\forall n$.

- $E_i\sigma_i = P_i(\sigma_i \geqslant 1) + P_i(\sigma_i \geqslant 2) + \dots = \sum_{\ell=0}^{\infty} P_i(\sigma_i > \ell) = \infty$.
  取M 使得$\sum_{\ell=0}^{M} P_i(\sigma_i > \ell) > \frac{1}{\varepsilon}$.
- $\{X_{n+M} = j\}$: $p_{ij}^{(n+M)} P_j(\sigma_j > 0)$
  $\{X_{n+M-1} = j, X_{n+M} \neq j\}$: $p_{ij}^{(n+M-1)} P_j(\sigma_j > 1)$
  $\{X_{n+M-2} = j, X_{n+M-1}, X_{n+M} \neq j\}$: $p_{ij}^{(n+M-2)} P_j(\sigma_j > 2)$
  $\dots$
  $\{X_n = j, X_{n+1}, \dots, X_{n+M} \neq j\}$: $p_{ij}^{(n)} P_j(\sigma_j > M)$
  $\{X_n, \dots X_{n+M} \neq j\}$
- 反证法: 若$* \geqslant \varepsilon$, 则$\sum$ RHS $\geqslant \varepsilon \times * > 1$, 矛盾!

证明, Claim 2. 当$n \gg 1$ 时, $|p_{ij}^{(n)} - p_{ij}^{(n+m)}| < \varepsilon, \forall 1 \leqslant m \leqslant M$.

- $\{Z_n = (W_n, Y_n)\}$ 不可约.

  情形I. $\{Z_n\}$ 非常返. 则

$$p_{ij}^{(n)} p_{ij}^{(n)} = \tilde{p}_{(i,i)(j,j)}^{(n)} \to 0.$$

- 情形II. $\{Z_n\}$ 常返. 取$\mu_j = 1_{\{j \neq i\}}, j \in S$;

$$\begin{aligned} |*| =& |P_{\mu \times \mu \mathbf{P}^m}(W_n = j) - P_{\mu \times \mu \mathbf{P}^m}(Y_n = j)| \\ \leqslant & 2P_{\mu \times \mu \mathbf{P}^m}(\tau > n) \to 0. \end{aligned}$$

总结: 假设不可约、非周期.

- 正常返: $p_{ij}^{(n)} \to \pi_j$.
- 零常返或非常返: $p_{ij}^{(n)} \to 0$.
- $P_\mu(X_n = j) \to \star$.

$$\sum_{i\in D} \mu_i p_{ij}^{(n)} + \sum_{i\notin D} \mu_i p_{ij}^{(n)} \approx \sum_{i\in D} \mu_i \star + \varepsilon \approx \sum_{i} \mu_i \star + 2\varepsilon.$$

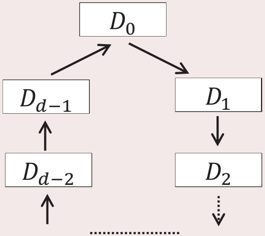
定理 (定理1.8.2)

假设不可约. 那么, 存在唯一的正整数d 以及S 的一个分割$D_0, D_1, \ldots, D_{d-1}$ (对任何n 补充定义$D_{nd+r} := D_r$), 使得:

- $\forall\ r \geqslant 0,\ i \in D_r,\ s \geqslant 0$,

  $\sum_{j \in D_{r+s}} p_{ij}^{(s)} = 1;$
- $\forall\ r \geqslant 0,\ i, j \in D_r,\ \exists\ N \geqslant 0$ 使得

  $p_{ij}^{(nd)} > 0,\ \forall n \geqslant N.$



- 定理中的d 被称为“周期”. 非周期指$d = 1$.
- $d > 1$ 时: $\hat{\mathbf{P}} = \mathbf{P}^d$ 在每个D_r 上不可约、非周期.

$$\mathbf{P}\ \text{正(零、非)常返iff}\ \hat{\mathbf{P}}\ \text{正(零、非)常返.}$$

总结: 假设不可约.

- 正常返、非周期:

$$p_{ij}^{(n)} \to \pi_j.$$

- 正常返、周期:

$$p_{ij}^{(nd+s)} \to d\pi_j$$
$$p_{ij}^{(m)} \to 0, \quad m \neq nd+s, \; i \in D_r, \; j \in D_{r+s}.$$

- 零常返:

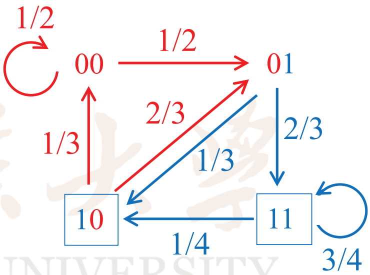
$$p_{ij}^{(n)} \to 0, \quad \text{但} \quad \sum_n p_{ij}^{(n)} = \infty.$$

- 非常返:

$$p_{ij}^{(n)} \to 0, \quad \text{且} \quad \sum_n p_{ij}^{(n)} < \infty.$$

§1.1 习题5. 投球成功的概率取决于前两次的投球成绩. 求成功概率、成功(频)率的渐近值.

- 成功概率: $p_n = P(\xi_n = 1)$
  $= P(X_n = 10) + P(X_n = 11)$.
- 强遍历: $\lim_n p_n = \pi_{10} + \pi_{11}$.
- $\frac{1}{2}\pi_{00} = \frac{1}{3}\pi_{10}$,
  $\frac{2}{3}\pi_{01} = \frac{1}{4}\pi_{11}$,
  $\pi_{10} = \pi_{01}$.
  解得$\pi = (\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{1}{2})$.
- 遍历定理: 成功频率$= \frac{1}{n}\sum_{m=0}^{n-1} 1_{\{\xi_m=1\}}$
  $= \frac{1}{n}\sum_{m=0}^{n-1}(1_{\{X_n=10\}} + 1_{\{X_n=01\}}) \to \pi_{10} + \pi_{11}$.



马氏链蒙特卡洛算法(Markov chain Monte Carlo, MCMC).

- 目标: 模拟$X \sim \pi$.
- 思想: 构造以π 为不变分布的马氏链转移矩阵$\mathbf{P}$.
- 输入初值i_0, 迭代产生$i_1, \ldots, i_n$.
- 输出 $X = i_n$ (强遍历定理).
- 处理$d \neq 1$: 考虑$\hat{\mathbf{P}} = \varepsilon\mathbf{I} + (1-\varepsilon)\mathbf{P}$.

  $\pi\mathbf{I} = \pi \Rightarrow \pi$ 为$\hat{\mathbf{P}}$ 的不变分布.

例:(Lawler书§7.3) 令$\mathbb{M}=\{\mathbf{M}=(m_{ij})_{N\times N}: m_{ij}=0$ 或$1\}$,

$$\mathcal{M}=\{\mathbf{M}\in\mathbb{M}: \text{若} m_{ij}=1 \text{ 则} m_{i\pm1,j}=m_{i,j\pm1}=0\}.$$

模拟服从$\mathcal{M}$ 上的均匀分布的"随机变量".

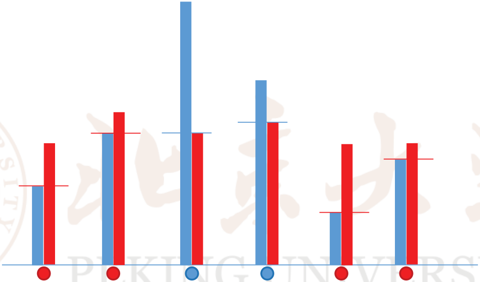
- $|\mathbb{M}|=2^{N^2}$, $|\mathcal{M}|\approx\beta^{N^2}$.
- 输入: $\mathbf{M}=\mathbf{0}$.
- 迭代: 任取(i,j), 改变m_{ij} 得到$\hat{\mathbf{M}}$.

  若$\hat{\mathbf{M}}\in\mathcal{M}$ 则用$\hat{\mathbf{M}}$ 代替$\mathbf{M}$.
- 迭代n 次后输出: $\mathbf{M}$.
- $\mathbf{M}\sim\hat{\mathbf{M}}$: 仅一个元素不同. 此时, $p_{\mathbf{M}\hat{\mathbf{M}}}=\frac{1}{N^2}$.

  $p_{\mathbf{MM}}=1-\frac{1}{N^2}\times\mathbf{M}$的邻居数.

§1.9 收敛速度

- 全变差距离: $d_{TV}(\mu,\nu) = \|\mu-\nu\| := \frac{1}{2}\sum_i |\mu_i-\nu_i|$.



- $= \mu(A)-\nu(A) = \nu(A)-\mu(A) = \sup_{A\subseteq S}|\mu(A)-\nu(A)|$.
- $= \inf_{X\sim\mu, Y\sim\nu} P(X\neq Y) = \inf_{\star\star} E1_{\{X\neq Y\}}$.
- $d_\varphi(\mu,\nu) := \inf_{\star\star} E\varphi(X,Y) = \sup_{|f(x)-f(y)|\leqslant\varphi(x,y)} (E_\mu f - E_\nu f)$.

- $\|\mu\mathbf{P}-\nu\mathbf{P}\| \leqslant \|\mu-\nu\|$.

  $$\begin{aligned}\text{LHS} &= \tfrac{1}{2}\textstyle\sum_j \left|\sum_i \mu_i p_{ij} - \sum_i \nu_i p_{ij}\right| \\ &= \tfrac{1}{2}\textstyle\sum_j \sum_i |\mu_i - \nu_i| p_{ij} = \text{RHS}.\end{aligned}$$

- 找 $X_0 \sim \mu,\ Y_0 \sim \nu$ 使得 $\text{RHS} = P(X_0 \neq Y_0)$. 取

  $$r_{(i,i)(k,k)} = p_{ik}; \quad r_{(i,j)(k,\ell)} = p_{ik}p_{j\ell},\ \forall i \neq j.$$

  则 $\text{LHS} \leqslant P(X_1 \neq Y_1) \leqslant P(X_0 \neq Y_0)$.

- $\|\mu\mathbf{P}^n - \pi\| \leqslant Ce^{-\beta n},\ \forall n \geqslant 0$.

- 例: $\mathbf{P} = \mathbf{P}^T$, $\& \cdots\cdots$, $\mu = \pi \bigoplus \pi^{\perp}$,

  $$\mu\mathbf{P}^n = \pi \bigoplus \lambda^n \pi^{\perp}, \quad |\lambda| < 1.$$

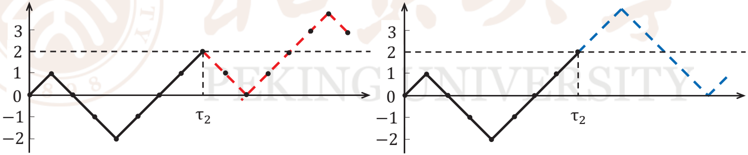
1. 首达时.

$$S_n = \xi_1 + \cdots + \xi_n, \quad P(\xi = 1) = P(\xi = -1) = \frac{1}{2}.$$

命题 (命题1.10.1, 反射原理)

$$P_0(\tau_i < n, S_n = i + j) = P_0(\tau_i < n, S_n = i - j).$$

- 前n 步轨道: $\{\tau_i < n, S_n = i + j\} \leftrightarrow \{\tau_i < n, S_n = i - j\}$.



- $\{X_n\}$ 仍然是SRW.

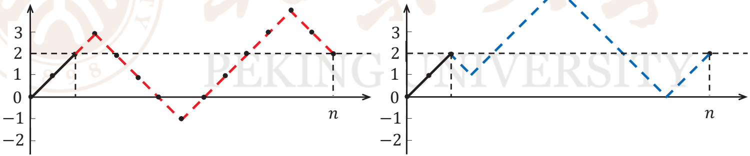
$$X_n = \begin{cases} S_n, & 若 n \leqslant \tau_i; \\ 2i - S_n, & 若 n > \tau_i \end{cases}$$

命题 (命题1.11.1)

$$P_0(\tau_i = n) = \frac{i}{n} P_0(S_n = i),\ \forall i \geqslant 1.$$

- $\tau_i = n$ 当且仅当$S_n = i$, $S_1, \ldots, S_{n-1} \neq i$.
- $\{S_n = i\} - \{\tau_i = n\} = \{\tau_i < n, S_n = i\}$.

$$P(A) = 2P(S_{n-1} = i + 1, S_n = i).$$

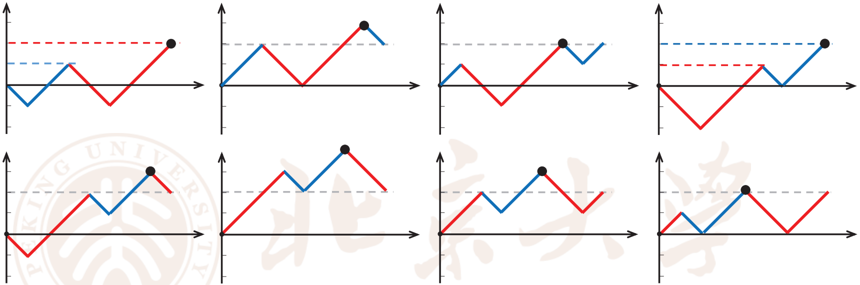


- $P(\tau_i = n) = P(S_n = i) - P(S_{n-1} = i + 1)$,

$$\star\star = C_{n-1}^{(n+i)/2} \frac{1}{2^{n-1}} = \frac{n-i}{n} P(S_n = i).$$

Ballot定理：假设$\xi \leqslant 1$，那么$P(\tau_i = n) = \frac{i}{n}P(S_n = i), \forall i \geqslant 1.$

- n 步轨道：$\omega, \theta\omega, \theta^2\omega, \theta^3\omega; \theta^4\omega, \theta^5\omega, \theta^6\omega, \theta^7\omega.$



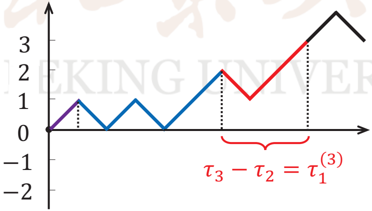
- $\frac{P(\tau_i=n)}{P(S_n=i)} = \frac{nP(\tau_i=n)}{n(p_1+\cdots+p_N)} = \frac{ip_1+\cdots ip_N}{n(p_1+\cdots+p_N)} = \frac{i}{n},$

p_1 :	ω_1	$\theta\omega_1$	$\theta^2\omega_1$	$\theta^3\omega_1$	$\theta^4\omega_1$	$\theta^5\omega_1$	$\theta^6\omega_1$	$\theta^7\omega_1$
p_2 :	ω_2	$\theta\omega_2$	$\theta^2\omega_2$	$\theta^3\omega_2$	$\theta^4\omega_2$	$\theta^5\omega_2$	$\theta^6\omega_2$	$\theta^7\omega_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
p_N :	ω_N	$\theta\omega_N$	$\theta^2\omega_N$	$\theta^3\omega_N$	$\theta^4\omega_N$	$\theta^5\omega_N$	$\theta^6\omega_N$	$\theta^7\omega_N$

- 推论1.10.3. $P(\tau_1 > 2n-1) = P(S_{2n} = 0)$

$$P(\tau_1 \leqslant 2n) = P(S_{2n} \geqslant 2) + P(S_{2n} \leqslant 0) = P(|S_{2n}| \geqslant 2).$$

- 推论1.10.3. $P_0(\tau_1 < \infty) = 1$, $E_0\tau_1 = \infty$. (零常返).
- 命题1.10.6. $\tau_i = \tau_1^{(1)} + \ldots + \tau_1^{(i)}$.



命题 (命题1.10.7. 反正弦律)

$$\lim_{n\to\infty} P_0(S_r \neq 0, \forall\ \delta n < r \leqslant n) = \frac{2}{\pi}\arcsin\sqrt{\delta}.$$

- 令$m = [\delta n]$. 则$\star = \sum_i P(S_m = i) P_i(\tau_0 > n - m)$.
- $1 - \star = \sum_i P(S_m = i) P_0(\tau_i \leqslant n - m)$.
- 取与$\{S_n\}$ 独立的SRW $\{T_n\}$:

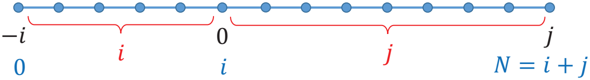
$$P_0(\tau_i \leqslant n - m) = P(T_{n-m} = i) + P(|T_{n-m}| > |i|).$$

- $1 - \star \approx \sum_i P(S_m = i) P(|T_{n-m}| > |i|) = P(|T_{n-m}| > |S_m|)$.
- $\star = P(|T_{n-m}| \leqslant |S_m|) \to \frac{2}{\pi}\arcsin\sqrt{\delta}$:

$$P\left(\sqrt{n-m}\cdot\left|\frac{T_{n-m}}{\sqrt{n-m}}\right| \leqslant \sqrt{m}\cdot\left|\frac{S_m}{\sqrt{m}}\right|\right) \approx P\left(\left|\frac{W}{Z}\right| \leqslant \frac{\sqrt{\delta}}{\sqrt{1-\delta}}\right).$$

3. Wald等式.

- $\tau := \tau_{-i} \wedge \tau_j$.



- $P_0(S_\tau = j) = P_i(\tau_N < \tau_0) = \frac{i}{i+j}$.
- $E_0 S_\tau = 0$:

$$E_0 S_\tau = j \cdot P_0(S_\tau = j) + (-i) \cdot P_0(S_\tau = -i) = 0.$$

- $E_0 \tau = E S_\tau^2$:

$$ES_\tau^2 = j^2 \cdot \frac{i}{i+j} + i^2 \cdot \frac{j}{i+j} = i \cdot j,$$
$$E_0 \tau = E_i(\tau_o \wedge \tau_N) = \sum_{j=1}^{N} E_i V_j = i \cdot j.$$

- 要点: $ES_n = 0$, $ES_n^2 = n$, $E_0 \tau^k < \infty$ (引理1.10.9).

$$S_n = \xi_1 + \cdots + \xi_n.$$

定理 (Wald 等式, 定理1.10.10)

假设τ 是$\{S_n\}$ 停时, $E|\xi|, E\tau < \infty$, 那么$ES_\tau = E\xi \cdot E\tau$.

- $S_\tau = \sum_{n=1}^{\tau} \xi_n = \sum_{n=1}^{\infty} \xi_n 1_{\{\tau \geqslant n\}}$.
- $\{\tau \leqslant n-1\} \in \sigma(S_1, \cdots, S_{n-1}) = \sigma(\xi_1, \ldots, \xi_{n-1})$.
- $|S_\tau| \leqslant \sum_{n=1}^{\infty} |\xi_n| 1_{\{\tau \geqslant n\}}$,

$$E|S_\tau| = \sum_{n=1}^{\infty} E|\xi_n| P_0(\tau \geqslant n) = E_0\tau \cdot E|\xi| < \infty.$$

- 因此,

$$ES_\tau = \sum_{n=1}^{\infty} E(\xi_n 1_{\{\tau \geqslant n\}}) = E\xi \cdot E\tau.$$

定理 (Wald 第二等式, 定理1.10.10)

假设τ 是$\{S_n\}$ 停时, $E\tau<\infty$, $E\xi=0$, $E\xi^2<\infty$, 存在M 使得$|S_n|\leqslant M$, $\forall n\leqslant\tau$. 那么, $ES_\tau^2=E\tau\cdot E\xi^2$.

- $S_{\tau\wedge N}=\sum_{n=1}^N\xi_n 1_{\{\tau\geqslant n\}}=\sum_{n=1}^{\tau\wedge N}\xi_n\xrightarrow{\text{a.s.}}S_\tau$.
- $S_{\tau\wedge N}^2=\sum_{n=1}^N\xi_n^2 1_{\{\tau\geqslant n\}}+2\sum_{1\leqslant n<m\leqslant N}\xi_n\xi_m 1_{\{\tau\geqslant m\}}\xrightarrow{\text{a.s.}}S_\tau^2$.
- $ES_{\tau\wedge N}^2=E_0(\tau\wedge N)\cdot E\xi^2$.
- 因为**, 所以$E_0S_{\tau\wedge N}^2\to ES_\tau^2$.
- 又有$E_0(\tau\wedge N)\to E_0\tau$.

补充知识:

- $\{S_n\}$ 是鞅: $E(S_{n+1}|\mathcal{F}_n) = S_n$.
- $\{S_n^2\}$ 是下鞅:

$$E(S_{n+1}^2|\mathcal{F}_n) = E(S_n^2 + 2\xi_{n+1}S_n + \xi_{n+1}^2|\mathcal{F}_n) = S_n^2 + 1 \geqslant S_n^2.$$

- $\{S_n^2 - n\}$ 是鞅.
- 选样定理: $\{M_n\}$ 是鞅, τ 是停时.

  若 $E|M_\tau| < \infty$ 且 $E|M_n|1_{\{\tau>n\}} \to 0$, 则 $EM_\tau = EM_0$.
- $\{S_n\}$: $|S_\tau| \leqslant N$, $E|S_n|1_{\{\tau>n\}} \leqslant NP(\tau > n) \to 0$.
- $\{S_n^2 - n\}$: $|S_\tau^2 - \tau| \leqslant N^2 + \tau$,

  $E|S_n^2 - n|1_{\{\tau>n\}} \leqslant E(N^2 + \tau)1_{\{\tau>n\}} \to 0$.

- 增量: $a_{n+1} := E(S_{n+1}^2|\mathcal{F}_n) - S_n^2 = E\xi^2 \in \mathcal{F}_n$
- 下鞅$\{S_n^2\}$ 的Doob 分解:

$$S_n^2 = M_n + A_n,$$

其中, $A_m = a_1 + \ldots + a_m = m$, $M_n = S_n^2 - A_n$, $\{M_n\}$ 是鞅.

- $\{M_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n\}$ 是鞅.
- $\tau = \tau_i \wedge \tau_{-i}$, $E\tau^2 < \infty$.
- $E|M_\tau| < \infty$:

$$|M_\tau| \leqslant i^4 + 5\tau i^2 + 3\tau^2 + 2\tau.$$

- $E|M_n|1_{\{\tau>n\}} \to 0$:

$$E|M_n|1_{\{\tau>n\}} \leqslant E(i^4 + 5\tau i^2 + 3\tau^2 + 2\tau)1_{\{\tau>n\}} \to 0.$$

- $EM_\tau = EM_0 = 0$:

$$i^4 - 6E\tau \cdot i^2 + 3E\tau^2 + 2E\tau = 0$$

- 因此, $E\tau^2 = \frac{1}{3}(5i^4 - 2i^2)$, $\quad \mathrm{Var}(\tau) = \frac{2}{3}i^2(i^2 - 1)$.

§1.11 分支过程(Branching process)

- $\{\xi_{ni} : n \geqslant 0; i \geqslant 1\}$ i.i.d.

  子代分布: $P(\xi = k) = p_k,\ k \in \mathbb{Z}_+$. (假设$p_1 \neq 1$)

- $X_0 = 1,\ X_1 = \xi_{01},\ X_{n+1} = \xi_{n1} + \ldots + \xi_{nX_n}$.

  $X_n = 0 \Rightarrow X_{n+1} = 0$.

- 分支树$\mathbb{T}$ (GW 树, Galton-Watson tree):

  $\xi = \xi_{01} = 4$;

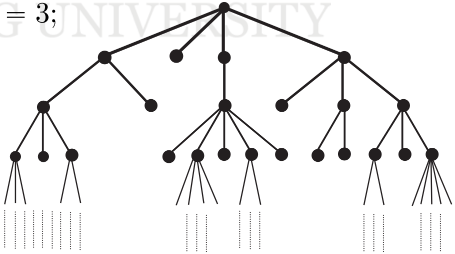
  $\xi_{11} = 2,\ \xi_{12} = 0,\ \xi_{13} = 1,\ \xi_{14} = 3$;

  $\xi_{21} = 3,\ \xi_{22} = 0,\ \xi_{23} = 5,$

  $\xi_{24} = 0,\ \xi_{25} = 2,\ \xi_{26} = 3$;

  $\xi_{31} = 3,\ \xi_{35} = 4,\ \xi_{3,13} = 5,$

  $\xi_{3i} = 2,\ i = 3, 7, 11, \ldots\ldots$



- 迭代结构I: $X_{n+1} = \xi_{n1} + \cdots + \xi_{nX_n}$.

$$E(X_{n+1}|\mathcal{F}_n) = E(\star\star|\mathcal{F}_n) = mX_n, \quad m = E\xi.$$

- $\{M_n = \frac{X_n}{m^n}\}$ 是鞅.
- $EX_n = m^n$. 若$m < 1$, 则

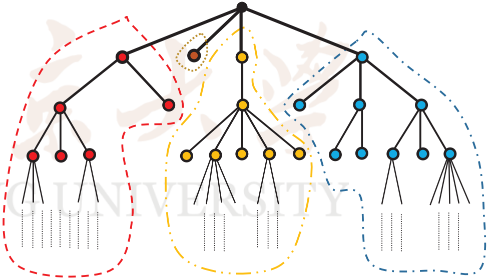
$$P(X_n \geqslant 1) \leqslant EX_n \to 0.$$

- 灭绝时间: $\tau_0 = \inf\{n \geqslant 0 : X_n = 0\}$.

  灭绝概率: $\rho = P_1(\tau_0 < \infty)$.
- 命题1.11.2. 若$m < 1$, 则

$$P(\tau_0 > n) \to 0 \quad \Rightarrow \quad \rho = 1.$$

- 迭代结构II: $X_{n+1} = X_n^{(1)} + \ldots + X_n^{(\xi)}$.



- $\mathbb{T}$ "=" $\mathbb{T}_1 \vee \ldots \vee \mathbb{T}_\xi$.

$$\rho = P(|\mathbb{T}| < \infty)$$
$$= P(|\mathbb{T}_1|, \ldots, |\mathbb{T}_\xi| < \infty).$$
$$= \sum_k p_k \rho^k.$$

- ξ 的母函数: $f(s) = f_\xi(s) = Es^\xi = \sum_k p_k s^k$.
  - $0 \leqslant s \leqslant 1 \Rightarrow 0 \leqslant f(s) \leqslant 1$.
  - $p_0 = f(0)$, $m = \sum_k kp_k = f'(1)$.
  - $f''(s) \geqslant 0$.
- X_n 的母函数:

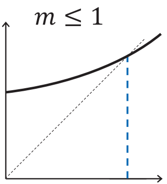
$$f_{X_{n+1}}(s) = Es^{X_{n+1}} = \sum_{k=0}^{\infty} P(X_n = k)E(s^{\xi_1+\ldots+\xi_k}|X_n = k)$$
$$= \sum_{k=0}^{\infty} P(X_n = k)f(s)^k = f_{X_n}(f(s)).$$

$$f_{X_n}(s) = f^{(n)}(s).$$

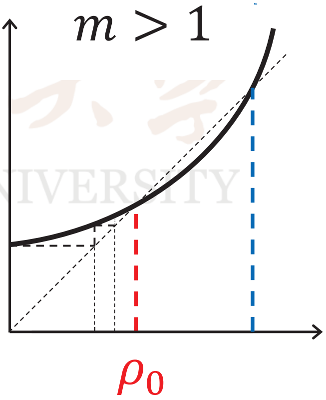
- $P(X_n = 0) = f^{(n)}(0)$.

命题1.11.3. $\rho = P_1(\tau_0 < \infty)$
是方程$s = f(s)$ 的最小非负解.

$\rho < 1 \Leftrightarrow m > 1.$



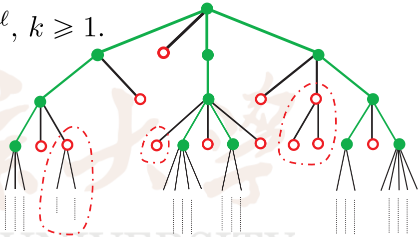
- $\rho = \lim_n P(\tau_0 \leqslant n)$
- $P(X_n = 0) = f_{X_n}(0)$
  $= f^{(n)}(0)$
  $\leqslant P(X_{n+1} = 0).$
- $f(\rho) = f(\lim_n f^{(n)}(0))$
  $= \lim_n f\left(f^{(n)}(0)\right)$
  $= \lim_n f^{(n+1)}(0) = \rho.$
- $f^{(n)}(0) \leqslant f^{(n)}(\rho_0) \Rightarrow \rho \leqslant \rho_0.$



相变(定义1.11.5): $m=\sum_k kp_k$, $\rho=P_1(\tau_0<\infty)$.

- 次(下)临界$m<1$: $\rho=1$.
    - $P_1(\lim_{n\to\infty}X_n=0)=1$.
- 临界$m=1$: $\rho=1$.
    - $P_1(\lim_{n\to\infty}X_n=0)=1$.
- 超(上)临界: $m>1$: $\rho<1$.
    - $\rho=P_1(\lim_{n\to\infty}X_n=0)<1$.
    - $1-\rho=P_1(\lim_{n\to\infty}X_n=\infty)>0$.

- $\tilde{\mathbb{T}} \sim \tilde{P}(\cdot) = P(\cdot \,|\, |\mathbb{T}| = \infty)$: $\tilde{m} = m$,
  $\tilde{p}_k = \sum_{\ell=0}^{\infty} p_{k+\ell} C_{k+\ell}^{k} (1-\rho)^{k-1}\rho^{\ell}$, $k \geqslant 1$.



- λ-biazed RW:
  $r_{u_{n-1}u_n} = \lambda^n$.
- $\frac{1}{R_\lambda} = \frac{1}{\lambda + \lambda R_\lambda^{(1)}} + \ldots + \frac{1}{\lambda + \lambda R_\lambda^{(\tilde{\xi})}}$.
  $R_\lambda = \infty$ iff $R_\lambda^{(i)} = \infty, \forall i$.
- $x = \tilde{P}(R_\lambda = \infty) = \sum_{k=1}^{\infty} \tilde{p}_k x^k = \tilde{f}(x)$.
- 结论: $\tilde{P}(R_\lambda = \infty) = 1$ 或0. (因为$\tilde{p}_0 = 0 \Rightarrow \tilde{\rho}_0 = 0$.)
- $\tilde{\mathbb{T}}$ 上的λ-biazed RW 常返iff $\lambda \geqslant m$.

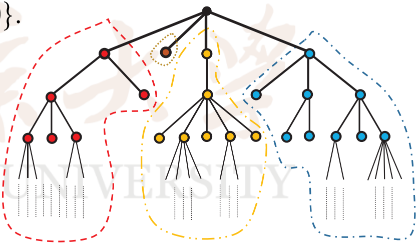
补充知识2: $m > 1$, $X_n \to \infty$ 的速度.

- 非负鞅 $M_n = \frac{X_n}{m^n} \xrightarrow{\text{a.s.}} W \geqslant 0$, 且 $EW < 1$.
- $P(W \in \mathbb{T}_0 \cup \mathbb{T}_+) = 1$, $\mathbb{T} = \mathbb{T}_0 \cup \mathbb{T}_+ \cup \mathbb{T}_\infty \cup \mathbb{T}_\nexists$,

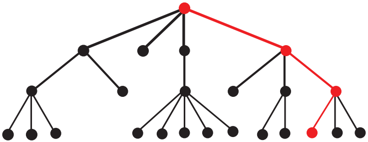
  $\mathcal{T}_0 = \{T : W(T) := \lim_n \frac{T_n}{m^n} = 0\}$.
- $\frac{X_{n+1}}{m^{n+1}} = \frac{1}{m}\left(\frac{X_n^{(1)}}{m^n} + \ldots + \frac{X_n^{(\xi)}}{m^n}\right)$.
- $W = \frac{1}{m}(W_1 + \ldots + W_\xi)$.

  $W = 0$ iff $W_i = 0$, $\forall i$.
- $P(W = 0) = p_0 + \sum_{k=1}^{\infty} p_k P(W = 0)^k$

  因此, $P(\mathcal{T}_0) = 1$ 或ρ.
- $P(\mathcal{T}_+) = 1 - \rho$ 或0. 在$\{|\mathbb{T}| = \infty\}$ 上, W 恒正 或$\equiv 0$.



带王位的分支过程: $(\hat{\mathbb{T}}, \Phi)$ spine.



- $\hat{p}_k = \frac{kp_k}{m}$, $k \geqslant 1$.
- $\hat{P}([\hat{\mathbb{T}}]_3 = [T]_3, [\Phi]_3 = [\phi]_3)$

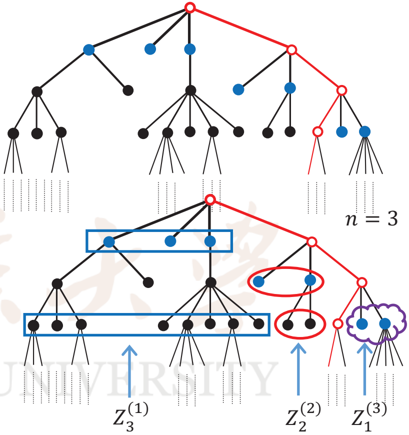
$$= \hat{p}_4 \cdot \frac{1}{4} \times p_2 p_0 p_1 \hat{p}_3 \cdot \frac{1}{3} \times p_3 p_0 p_5 p_0 p_0 \hat{p}_3 \cdot \frac{1}{3}$$

- $\hat{P}([\hat{\mathbb{T}}]_3 = [T]_3, [\Phi]_3 = [\phi]_3) = \frac{1}{m^3} P([\mathbb{T}]_3 = [T]_3)$
- $\hat{P}([\hat{\mathbb{T}}]_3 = [T]_3) = T_3 \times \frac{1}{m^3} P([\mathbb{T}]_3 = [T]_3)$.
- $\hat{P}([\hat{\mathbb{T}}]_n = [T]_n) = P([\mathbb{T}]_n = [T]_n) \times \frac{T_n}{m^n}$.
- $d\hat{P}(T) = dP(T) \times W$. "因此",

$$P(\mathcal{T}_+) > 0 \text{ iff } \hat{P}(\mathcal{T}_+) > 0; \qquad \hat{P}(\mathcal{T}_0) = 0.$$

$\hat{P}(\cdot)$ 是带移民的分支过程:

- $P(Y=k)=\hat{p}_{k+1}$.
- $Z_{n+1}=\xi_{n1}+\ldots+\xi_{nZ_n}+Y_{n+1}$.
- $\{\frac{Z_n}{m^n}\}$ 是$E(\cdot|\mathcal{Y})$ 下鞅.
- $Z_n=Z_n^{(1)}+Z_{n-1}^{(2)}+\ldots+Z_1^{(n)}$.
- $E(Z_n|\mathcal{Y})$
  $=m^{n-1}Y_1+\ldots+mY_{n-1}+Y_n$.
- $E(\frac{Z_n}{m^n}|\vec{Y}=\vec{y})=\frac{y_1}{m}+\ldots+\frac{y_n}{m^n}$.
- $E\ln^+ Y<\infty$ $(\sum_{k\geqslant 2}kp_k\ln k<\infty)$:
  $\sum_{k=1}^\infty \frac{Y_k}{m^k}\overset{\text{a.s.}}{<}\infty \Rightarrow W=\lim_n \frac{Z_n}{m^n}$ 存在且有限. $\hat{P}(\mathbb{T}_+)=1$.
- $E\ln^+ Y=\infty \Rightarrow \limsup_n \frac{Z_n}{m^n}=\infty$, a.s.. $\hat{P}(\mathbb{T}_+)=0$.



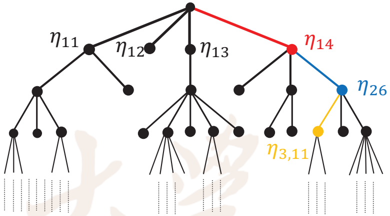
补充知识3. 分支随机游动(BRW)

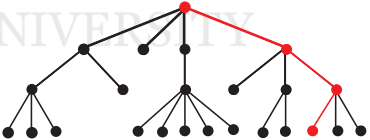
- η_{ni}, i.i.d., e.g. $P(\eta=\pm 1)=\frac{1}{2}$.
- $S_u=\eta_{u_0}+\ldots+\eta_{u_n}$, $u\in\mathbb{T}_n$.
- $M_n=\max_{u\in\mathbb{T}_n} S_u$. 迭代结构:

$$M_{n+1}=\max_{1\leqslant i\leqslant\xi}(\eta_i+M_n^{(i)}).$$

- $V_1=e^{\eta_1}+\ldots+e^{\eta_\xi}$, $\mu=\mathbb{E}V_1$.
- 令 $V_n=\sum_{u\in\mathbb{T}_n} e^{S(u)}$,
  则 $\{\frac{V_n}{\mu^n}\}$ 是鞅. $EV_n=\mu^n$.
- $\mathbb{E}V_n=m^n\hat{\mathbb{E}}e^{S(\phi_n)}$.
- $M_n/n\overset{\text{a.s.}}{\to}x^*>0$. 例: $\xi\equiv d$, x^* 的上估计: $\forall a>0$,

$$P(\frac{M_n}{n}>y)\leqslant d^n\cdot P(S_n>ny)\leqslant\left(d\cdot Ee^{a(\eta-y)}\right)^n.$$



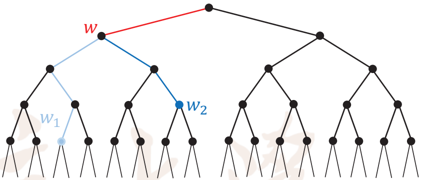


$\xi \equiv d.$

- $ES_{w_1}S_{w_2} = |w| = n$:

  $S_{w_1} = \eta_1 + \cdots + \eta_w + \sum_i \hat{\eta}_i$,

  $S_{w_2} = \star + \sum_j \tilde{\eta}_j$.



- 考虑前N 层树上的

  随机游动的格林函数

  $G_{w,v} := E_w \sum_{n=0}^{\tau_o - 1} 1_{\{X_n = v\}}$.

$$G_{w_1,w_2} = G_{w,w_2} = G_{w_2,w} = G_{w,w} = a_n.$$

- $a_n = (d+1)n$.

$$a_n = 1 + \frac{d}{d+1} a_n + \frac{1}{d+1} a_{n-1} \Rightarrow a_n = a_{n-1} + (d+1).$$