Supplementary Material for "Policy Evaluation for Temporal and/or Spatial Dependent Experiments"

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S.1. Algorithms, Assumptions and Lemmas

Let $\widetilde{\boldsymbol{V}}_\theta(\tau_1,\tau_2)$ and $\boldsymbol{V}_{\widetilde{\theta}}(\tau_1,\tau_2)$ be the submatrices of $\widetilde{\boldsymbol{V}}_\theta$ and $\boldsymbol{V}_{\widetilde{\theta}}$, respectively, formed by rows in $\{(\tau_1-1)(d+2)+1,(\tau_1-1)(d+2)+2,\dots,\tau_1(d+2)\}$ and columns in $\{(\tau_2-1)(d+2)+1,(\tau_2-1)(d+2)+2,\dots,\tau_2(d+2)\}$. We first introduce some auxiliary lemmas.

Lemma S.1. *Under TCMIA and Assumptions 3 - 4, as $n,m\to\infty$, $h\to 0$, and $mh\to\infty$, we have* $\sup_{\tau_1,\tau_2}|\widetilde{\boldsymbol{V}}_\theta(\tau_1,\tau_2)-\boldsymbol{V}_{\widetilde{\theta}}(\tau_1,\tau_2)|=o_p(1)$.

Lemma S.2. *Under STCMIA, Assumptions 3 and 6, as $n,m,r\to\infty$, $h,h_{st}\to 0$ and $mh, rh_{st}\to\infty$, then* $\sup_{\tau_1,\iota_1,\tau_2,\iota_2}|\widetilde{\boldsymbol{V}}_{\theta,st}(\tau_1,\iota_1,\tau_2,\iota_2)-\boldsymbol{V}_{\widetilde{\theta}_{st}}(\tau_1,\iota_1,\tau_2,\iota_2)|=o_p(1)$.

We describe our inference procedure for DE under the spatio-temporal case here. A pseudocode summarizing our algorithm is given in Algorithm S.1. We denote for $\iota=1,\dots,r$,

$$\begin{aligned}\boldsymbol{Y}_i&=\mathrm{diag}\{Y_{i,1,1},\dots,Y_{i,m,1},\dots,Y_{i,1,r},\dots,Y_{i,m,r}\},\\ \boldsymbol{Z}_i&=\mathrm{diag}\{Z_{i,1,1}^\top,\dots,Z_{i,m,1}^\top,\dots,Z_{i,1,r}^\top,\dots,Z_{i,m,r}^\top\}.\end{aligned}\tag{S.1}$$

Denote the longitude and latitude (scaled to be $[0,1]$) of region ι by (u_ι,v_ι),

$$\kappa_{\ell,h_{st}}(\iota)=\frac{K\{(u_\iota-u_\ell)/h_{st}\}K\{(v_\iota-v_\ell)/h_{st}\}}{\sum_{j=1}^r K\{(u_\iota-u_j)/h_{st}\}K\{(v_\iota-v_j)/h_{st}\}}.\tag{S.2}$$

Let $\mathcal{K}=\mathcal{K}_1\mathcal{K}_2$, where $\mathcal{K}_1$ is a block matrix whose (ι,ℓ)th block is $\kappa_{\ell,h_{st}}(\iota)\boldsymbol{J}_{pm}$ for $1\le\iota,\ell\le r$ and $\mathcal{K}_2=\mathrm{diag}\{\Omega,\dots,\Omega\}$. The estimation and inference procedure of DE in the spatio-temporal case is given as follows.

Algorithm S.1 Inference of DE under the spatio-temporal design

1: Compute $\widehat{\theta}_{st}^0(\tau,\iota)=\left(\sum_{i=1}^n Z_{i,\tau,\iota}^\top Z_{i,\tau,\iota}\right)^{-1}\left(\sum_{i=1}^n Z_{i,\tau,\iota}^\top Y_{i,\tau,\iota}\right)$ and $\widetilde{\theta}_{st}^0(\tau,\iota)=\sum_{j=1}^m\omega_{j,h}(\tau)\widehat{\theta}(j,\iota)$ for each τ,ι.
2: Compute $\widetilde{\theta}_{st}(\tau,\iota)=\sum_{\ell=1}^r\kappa_{\ell,h_{st}}(\iota)\widetilde{\theta}(\tau,\ell)$.
3: Estimate the covariance Σ_y by the following steps:
  (i). estimate the combined noise by $\widehat{e}_{i,\tau,\iota}=Y_{i,\tau,\iota}-Z_{i,\tau,\iota}^\top\widetilde{\theta}_{st}(\tau,\iota)$;
  (ii). estimate the subject effects and measurement errors by

$$\begin{aligned}\widehat{\eta}_{i,\tau,\iota}^I=\textstyle\sum_{\ell=1}^r\kappa_{\ell,h_{st}}(\iota)\sum_{j=1}^m\omega_{j,h}(\tau)\widehat{e}_{i,j,\ell},\quad \widehat{\eta}_{i,\tau,\iota}^{II}=\sum_{\ell=1}^r\kappa_{\ell,h_{st}}(\iota)\widehat{e}_{i,j,\ell}-\widehat{\eta}_{i,\tau,\iota}^I,\\ \widehat{\eta}_{i,\tau,\iota}^{III}=\textstyle\sum_{j=1}^m\omega_{j,h}(\tau)\widehat{e}_{i,j,\ell}-\widehat{\eta}_{i,\tau,\iota}^I,\quad \widehat{\varepsilon}_{i,\tau,\iota}=\widehat{e}_{i,\tau,\iota}-\widehat{\eta}_{i,\tau,\iota}^I-\widehat{\eta}_{i,\tau,\iota}^{II}-\widehat{\eta}_{i,\tau,\iota}^{III}.\end{aligned}\tag{S.3}$$

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(iii). the covariances of η and ε are estimated by

$$\begin{aligned} \widehat{\Sigma}_{\eta^I}(\tau_1,\iota_1,\tau_2,\iota_2) = \tfrac{1}{n-1}\sum_{i=1}^n \widehat{\eta}^I_{i,\tau_1,\iota_1}\widehat{\eta}^I_{i,\tau_2,\iota_2}, \quad \widehat{\Sigma}_{\eta^{II}}(\tau_1,\iota_1,\tau_2) = \tfrac{1}{n-1}\sum_{i=1}^n \widehat{\eta}^{II}_{i,\tau_1,\iota_1}\widehat{\eta}^{II}_{i,\tau_2,\iota_1}, \\ \widehat{\Sigma}_{\eta^{III}}(\tau_1,\iota_1,\iota_2) = \tfrac{1}{n-1}\sum_{i=1}^n \widehat{\eta}^{III}_{i,\tau_1,\iota_1}\widehat{\eta}^{III}_{i,\tau_1,\iota_2}, \quad \widehat{\sigma}^2_{\varepsilon}(\tau_1,\iota_1) = \tfrac{1}{n-1}\sum_{i=1}^n \widehat{\varepsilon}^2_{i,\tau_1,\iota_1}; \end{aligned} \tag{S.4}$$

(iv). the covariance of outcome is estimated by

$$\begin{aligned} \widehat{\Sigma}_y(\tau_1,\iota_1,\tau_2,\iota_2) = & \ \widehat{\Sigma}_{\eta^I}(\tau_1,\iota_1,\tau_2,\iota_2) + \widehat{\Sigma}_{\eta^{II}}(\tau_1,\iota_1,\tau_2)I(\iota_1=\iota_2) \\ & +\widehat{\sigma}^2_{\varepsilon^I}(\tau_1,\iota_1,\iota_2)I(\tau_1=\tau_2) + \widehat{\sigma}^2_{\varepsilon^{II}}(\tau_1,\iota_1)I(\tau_1=\tau_2,\iota=\iota_2). \end{aligned}$$

4: Compute

$$\widehat{\boldsymbol{V}}_{\theta_{st}} = \left\{\sum_{i=1}^n \boldsymbol{Z}_i^\top \boldsymbol{Z}_i\right\}^{-1} \left\{\sum_{i=1}^n \boldsymbol{Z}_i^\top \widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{Z}_i\right\} \left\{\sum_{i=1}^n \boldsymbol{Z}_i^\top \boldsymbol{Z}_i\right\}^{-1}$$

where $\widehat{\boldsymbol{\Sigma}} = \{\widehat{\Sigma}_y(\tau_1,\iota_1,\tau_2,\iota_2)\}_{\tau_1,\iota_1,\tau_2,\iota_2}$ and $\widetilde{\boldsymbol{V}}_{\theta_{st}} = \mathcal{K}\widehat{\boldsymbol{V}}_{\theta_{st}}\mathcal{K}^\top$.

5: Calculate $\widehat{\text{DE}}_{st}$ and the standard error $\widehat{se}(\widehat{\text{DE}}_{st})$ based on $\widetilde{\boldsymbol{V}}_{\theta_{st}}$.

6: Reject H_0^{DE} if $\widehat{\text{DE}}_{st}/\widehat{se}(\widehat{\text{DE}}_{st})$ exceeds the upper αth quantile of a standard normal distribution.

Algorithm S.2 Inference of IE under the spatio-temporal design

1: Compute the OLS estimator

$$\widehat{\boldsymbol{\Theta}} = \left\{\sum_{i=1}^n \boldsymbol{Z}_{i,(-m)}\boldsymbol{Z}_{i,(-m)}^\top\right\}^{-1} \left\{\sum_{i=1}^n \boldsymbol{Z}_{i,(-m)}\boldsymbol{S}_{i,(-1)}^\top\right\}.$$

2: Compute $\widetilde{\Theta}_{st} = \mathcal{K}\widehat{\Theta}$.

3: Plug-in the parameter estimates $\widetilde{\Theta}_{st}$ and $\widetilde{\theta}_{st}$ to obtain $\widehat{\text{IE}}_{st}$.

4: Compute the residuals $\widehat{E}_{i,\tau,\iota} = S_{i,\tau,\iota} - Z_{i,\tau,\iota}^\top\widetilde{\Theta}(\tau,\iota)$.

5: **for** $b = 1,\ldots,B$ **do**

generate i.i.d. standard normal random variables $\{\xi_i^b\}_{i=1}^n$;

generate pseudo outcomes $S^b_{i,\tau,\iota}$ and $Y^b_{i,\tau,\iota}$ by $S^b_{i,\tau+1,\iota} = Z_{i,\tau,\iota}\widetilde{\Theta}(\tau,\iota)+\xi_i\widehat{E}_{i,\tau,\iota}$ and $Y^b_{i,\tau,\iota} = Z_{i,\tau,\iota}\widetilde{\theta}_{st}(\tau,\iota)+\xi\widehat{e}_{i,\tau,\iota}$, where $Z^b_{i,\tau,\iota} = \{1,(S^b_{i,\tau,\iota})^\top, A_{i,\tau,\iota},\bar{A}_{i,\tau,\mathcal{N}_\iota}\}^\top$;

substitute $Y_{i,\tau,\iota}$ and $S_{i,\tau,\iota}$ with $Y^b_{i,\tau,\iota}$ and $S^b_{i,\tau,\iota}$, and repeat the procedures in Steps 1-3 to obtain the plug-in estimator $\widehat{\text{IE}}^b_{st}$.

6: **end for**

7: Reject H_0^{IE} if $\widehat{\text{IE}}_{st}$ exceeds the upper αth empirical quantile of $\{\widehat{\text{IE}}^b_{st} - \widehat{\text{IE}}_{st}\}_b$.

S.2. Proof of Lemma 1

We first prove (4). It follows from the law of total expectation that

$$\mathbb{E}(Y_\tau|\bar{A}_\tau,\bar{S}_\tau) = \mathbb{E}^{\bar{Y}_{\tau-1}|\bar{A}_\tau,\bar{S}_\tau}\{\mathbb{E}(Y_\tau|\bar{A}_\tau,\bar{S}_\tau,\bar{Y}_{\tau-1})\},$$

where $\bar{A}_\tau$, $\bar{S}_\tau$ and $\bar{Y}_{\tau-1}$ denote the history of actions, states and outcomes, respectively. The first expectation on the right-hand-side (RHS) is taken with respect to the conditional distribution of $\bar{Y}_{\tau-1}$ given that $(\bar{A}_\tau,\bar{S}_\tau)$.

Without loss of generality, assume both the outcome and the state are discrete. Let $p^{\bar{Y}_{\tau-1}|\bar{A}_\tau,\bar{S}_\tau}$ denotes the conditional probability mass function of $\bar{Y}_{\tau-1}$ given $\bar{A}_\tau,\bar{S}_\tau$, we have

$$\mathbb{E}(Y_\tau|\bar{A}_\tau=\bar{a}_\tau,\bar{S}_\tau=\bar{s}_\tau) = \sum_{\bar{y}_{\tau-1}} p^{\bar{Y}_{\tau-1}|\bar{A}_\tau=\bar{a}_\tau,\bar{S}_\tau=\bar{s}_\tau}(\bar{y}_{\tau-1})\{\mathbb{E}(Y_\tau|\bar{A}_\tau=\bar{a}_\tau,\bar{S}_\tau=\bar{s}_\tau,\bar{Y}_{\tau-1}=\bar{y}_{\tau-1})\}.$$

According to CA, the second term on the RHS is equal to

$$\mathbb{E}[Y^*_\tau(\bar{a}_\tau)|\bar{A}_\tau=\bar{a}_\tau,\bar{S}^*_\tau(\bar{a}_{\tau-1})=\bar{s}_\tau,\bar{Y}^*_{\tau-1}(\bar{a}_{\tau-1})=\bar{y}_{\tau-1}],$$

where $\bar{S}^*_\tau(\bar{a}_{t-1})$ and $\bar{Y}^*_{\tau-1}(\bar{a}_{\tau-1})$ denote the sets of potential states and outcomes up to time τ and $\tau-1$, respectively. It follows that

$$\mathbb{E}(Y_\tau|\bar{A}_\tau=\bar{a}_\tau,\bar{S}_\tau=\bar{s}_\tau)=\sum_{\bar{y}_{\tau-1}} p^{\bar{Y}_{\tau-1}|\bar{A}_\tau=\bar{a}_\tau,\bar{S}_\tau=\bar{s}_\tau}(\bar{y}_{\tau-1})\{\mathbb{E}[Y^*_\tau(\bar{a}_\tau)|\bar{A}_\tau=\bar{a}_\tau,\bar{S}^*_\tau(\bar{a}_{\tau-1})=\bar{s}_\tau,\bar{Y}^*_{\tau-1}(\bar{a}_{\tau-1})=\bar{y}_{\tau-1}]\}.$$

Under SRA and PA, the conditional expectation on the right-hand-side is independent of the actions. In addition, it is equal to $R_\tau(\bar{a}_\tau,\bar{s}_\tau)$, independent of $\bar{y}_{\tau-1}$. This yields (4).

We next show (5). Using similar arguments, we can show that

$$\mathbb{E}\{R_\tau(a_\tau,S^*_\tau(\bar{a}_{\tau-1}),\cdots,S_1)\}=\mathbb{E}[\mathbb{E}\{R_\tau(a_\tau,S^*_\tau(\bar{a}_{\tau-1}),\cdots,S_1)|A_1=a_1,\bar{S}^*_{\tau-1}(\bar{a}_{\tau-2}),\bar{Y}^*_{\tau-1}(\bar{a}_{\tau-1})\}].$$

Under CA, we can replace $Y^*_1(a_1)$ and $S^*_2(a_1)$ with Y_1 and S_2, respectively. Under SRA and PA, the event $A_2=a_2$ can be included in the conditioning set. This yields that

$$\begin{aligned}&\mathbb{E}\{R_\tau(a_\tau,S^*_\tau(\bar{a}_{\tau-1}),\cdots,S_1)\}\\=\;&\mathbb{E}[\mathbb{E}\{R_\tau(a_\tau,S^*_\tau(\bar{a}_{\tau-1}),\cdots,A_1,S_1)|A_2=a_2,A_1=a_1,\bar{S}^*_{\tau-1}(\bar{a}_{\tau-2}),\bar{Y}^*_{\tau-1}(\bar{a}_{\tau-1}),S_1,Y_1\}].\end{aligned}$$

Iteratively applying this argument allows us to repeatedly replace the counterfactual variables with the observed ones. At the end, all the potential outcomes/states will be replaced with the observed versions conditional on the actions. The proof is hence completed.

S.3. Proof of Proposition 1

Recall that

$$\begin{aligned}\text{DE}&=\sum_{\tau=1}^m\mathbb{E}\{R_\tau(1,S^*_\tau(\mathbf{0}_{\tau-1}),0,S^*_{\tau-1}(\mathbf{0}_{\tau-2}),\ldots,S_1)-R_\tau(0,S^*_\tau(\mathbf{0}_{\tau-1}),0,S^*_{\tau-1}(\mathbf{0}_{\tau-2}),\ldots,S_1)\},\\ \text{IE}&=\sum_{\tau=1}^m\mathbb{E}\{R_\tau(1,S^*_\tau(\mathbf{1}_{\tau-1}),1,S^*_{\tau-1}(\mathbf{1}_{\tau-2}),\ldots,S_1)-R_\tau(1,S^*_\tau(\mathbf{0}_{\tau-1}),0,S^*_{\tau-1}(\mathbf{0}_{\tau-2}),\ldots,S_1)\}.\end{aligned}$$

Under Model 1, each summand in DE equals $\gamma(\tau)$. It follows that

$$\text{DE}=\sum_{\tau=1}^m\gamma(\tau).$$

Similarly, for IE, we have

$$\begin{aligned}&\mathbb{E}\{R_\tau(1,S^*_\tau(\mathbf{1}_{\tau-1}),1,S^*_{\tau-1}(\mathbf{1}_{\tau-2}),\ldots,S_1)-R_\tau(1,S^*_\tau(\mathbf{0}_{\tau-1}),0,S^*_{\tau-1}(\mathbf{0}_{\tau-2}),\ldots,S_1)\}\\ =&\mathbb{E}\{\beta_0(\tau)+S^*_\tau(\mathbf{1}_{\tau-1})^\top\beta(\tau)+\gamma(\tau)\}-\mathbb{E}\{\beta_0(\tau)+S^*_\tau(\mathbf{0}_{\tau-1}))^\top\beta(\tau)+\gamma(\tau)\}\\ =&\mathbb{E}\{S^*_\tau(\mathbf{1}_{\tau-1})-S^*_\tau(\mathbf{0}_{\tau-1})\}^\top\beta(\tau)\\ =&\mathbb{E}[\Phi(\tau-1)\{S^*_{\tau-1}(\mathbf{1}_{\tau-2})-S^*_{\tau-1}(\mathbf{0}_{\tau-2})\}+\Gamma(\tau-1)]^\top\beta(\tau)\\ =&\mathbb{E}[\Phi(\tau-1)\Phi(\tau-2)\{S^*_{\tau-2}(\mathbf{1}_{\tau-3})-S^*_{\tau-2}(\mathbf{0}_{\tau-3})\}+\Phi(\tau-1)\Gamma(\tau-2)+\Gamma(\tau-1)]^\top\beta(\tau)\\ \cdots&\\ =&\beta(\tau)^\top\left\{\sum_{k=1}^{\tau-1}\left(\prod_{l=k+1}^{\tau-1}\Phi(l)\right)\Gamma(k)\right\},\end{aligned}$$

which completes the proof.

S.4. Proofs of Lemmas S.1 and S.2

The proof of Lemma S.2 is similar to that of Lemma S.1. Hence, we focus on proving Lemma S.1 for space economy.

Proof: We first prove that $\sup_{\tau_1,\tau_2}|\widehat{\Sigma}_y(\tau_1,\tau_2)-\Sigma_y(\tau_1,\tau_2)|=o_p(1)$. It suffices to show that $n^{-1}\sum_{i=1}^n\widehat{\eta}_{i,\tau_1}\widehat{\eta}_{i,\tau_2}$ and $n^{-1}\sum_{i=1}^n\widehat{\varepsilon}^2_{i,\tau}$ are consistent estimators of $\Sigma_\eta(\tau_1,\tau_2)$ and $\sigma^2_{\varepsilon,\tau}$. According to Section 2.2, we have $\widehat{e}_{i,\tau}=Y_{i,\tau}-Z_{i,\tau}^\top\widetilde{\theta}(\tau)$. Notice that

$$\widehat{\eta}_{i,\tau}=\sum_{j=1}^m\omega_{j,h}(\tau)\widehat{e}_i(j).$$

We follow notations in Zhu et al. (2014) and write

$$\bar{\varepsilon}_{i,\tau} = \sum_{j=1}^{m} \omega_{j,h}(\tau)\varepsilon_{i,j}, \quad \Delta_K \eta_{i,\tau} = \sum_{j=1}^{m} \omega_{j,h}(\tau)\left\{\eta_{i,j} - \eta_{i,\tau}\right\},$$

$$\Delta_K \theta(\tau) = \sum_{j=1}^{m} \omega_{j,h}(\tau)\left\{\theta(j) - \widehat{\theta}(j)\right\}, \quad \Delta_{\eta_i}(\tau) = \bar{\varepsilon}_{i,\tau} + \Delta_K \eta_{i,\tau} + Z_{i,\tau}^{\top} \Delta_K \theta(\tau).$$

Then we have

$$\widehat{\eta}_{i,\tau} - \eta_{i,\tau} = \Delta_{\eta_i}(\tau),$$

which gives

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} \widehat{\eta}_{i,\tau_1}\widehat{\eta}_{i,\tau_2} &= n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\eta_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} \Delta_{\eta_i}(\tau_1)\Delta_{\eta_i}(\tau_2) \\ &\quad + n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\Delta_{\eta_i}(\tau_2) + n^{-1}\sum_{i=1}^{n} \Delta_{\eta_i}(\tau_1)\eta_{i,\tau_2}. \end{aligned}$$

The first term $n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\eta_{i,\tau_2}$ converges to $\Phi_\eta(\tau_1, \tau_2)$ according to the Law of Large Number. We next show

(a) $I_1 = n^{-1}\sum_{i=1}^{n} \Delta_{\eta_i}(\tau_1)\Delta_{\eta_i}(\tau_2)$ converges to zero for any $(\tau_1, \tau_2) \in \mathcal{T}^2$.

(b) $I_2 = n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\Delta_{\eta_i}(\tau_2) + n^{-1}\sum_{i=1}^{n} \Delta_{\eta_i}(\tau_1)\eta_{i,\tau_2}$ converges to zero for any $(\tau_1, \tau_2) \in \mathcal{T}^2$.

By mutually multiplying the three terms in the summation form of $\Delta_{\eta_i}(\tau)$, we have

$$\begin{aligned} I_1 = {} & n^{-1}\sum_{i=1}^{n} \bar{\varepsilon}_{i,\tau_1}\bar{\varepsilon}_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} \Delta_K\eta_{i,\tau}\Delta_K\eta_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} Z_{i,\tau_1}^{\top}\Delta_K\theta(\tau_1)\Delta_K\theta(\tau_2)^{\top} Z_{i,\tau_2} \\ & + n^{-1}\sum_{i=1}^{n} \bar{\varepsilon}_{i,\tau_1}\Delta_K\eta_i(\tau_2) + n^{-1}\sum_{i=1}^{n} \Delta_K\eta_{i,\tau_1}\bar{\varepsilon}_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} \bar{\varepsilon}_{i,\tau_1}\Delta_K\theta(\tau_2)^{\top} Z_{i,\tau_2} \\ & + n^{-1}\sum_{i=1}^{n} Z_{i,\tau_1}^{\top}\Delta_K\theta(\tau_1)\bar{\varepsilon}_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} \Delta_K\eta_{i,\tau_1}\Delta_K\theta(\tau_2)^{\top} Z_{i,\tau_2} + n^{-1}\sum_{i=1}^{n} Z_{i,\tau_1}^{\top}\Delta_K\theta(1)\Delta_K\eta_{i,\tau_2} \end{aligned}$$

By the independence between ε_{i,τ_1} and ε_{i,τ_2}, the first term $n^{-1}\sum_{i=1}^{n} \bar{\varepsilon}_{i,\tau_1}\bar{\varepsilon}_{i,\tau_2}$ converges to zero. As for the second term, using standard arguments in establishing theoretical properties of kernel estimators‡, the bias term satisfies $\mathbb{E}\sum_{j=1}^{m} \omega_{j,h}(\tau)\left\{\eta_{i,j} - \eta_{i,\tau}\right\} = O(h^2 + m^{-1})$, whereas the variance term satisfies $\mathrm{Var}[\sum_{j=1}^{m} \omega_{j,h}(\tau)\left\{\eta_{i,j} - \eta_{i,\tau}\right\}] = O(m^{-1}h^{-1})$. It follows that

$$\begin{aligned} & n^{-1}\sum_{i=1}^{n} \Delta_K\eta_{i,\tau_1}\Delta_K\eta_{i,\tau_2} \\ = & n^{-1}\sum_{i=1}^{n} \left[\sum_{j=1}^{m} \omega_{j,h}(\tau_1)\left\{\eta_{i,j} - \eta_{i,\tau_1}\right\}\right]\left[\sum_{j=1}^{m} \omega_{j,h}(\tau_2)\{\eta_{i,j} - \eta_{i,\tau_2}\}\right] \\ = & O_p(h^4 + m^{-1}h^{-1}). \end{aligned}$$

As for the third term, notice that $\left\{\widehat{\theta}(\tau) - \theta(\tau) : \tau\right\}$ converges uniformly to zero, $\{\Delta_K\theta(\tau) : \tau\}$ converges uniformly to zero as well. Under the given conditions, $n^{-1}\sum_{i=1}^{n} Z_{i,\tau_1}^{\top} Z_{i,\tau_2}$ is $O_p(1)$. It follows that the third term is $o_p(1)$. The remaining six cross products converges to zero according to the Law of Large Number and the mutual independence of Z_i, ε_i, and η_i imposed in Assumption 4 . This completes the proof of (a).

To prove (b), we only need to prove $n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\Delta_K\eta_{i,\tau_2} = o_p(1)$ since that η_i is independent of Z_i and ε_i. This follows from the fact that

$$n^{-1}\sum_{i=1}^{n} \eta_{i,\tau_1}\left[\sum_{j=1}^{m} \omega_{j,h}(\tau_2)\{\eta_{i,j} - \eta_{i,\tau_2}\}\right]$$

‡See e.g., `http://www.stat.cmu.edu/~larry/=sml/NonparRegression.pdf`.

$$=\sum_{j=1}^{m}\omega_{j,h}(\tau_2)n^{-1}\left\{\sum_{i=1}^{n}\eta_{i,j}\eta_{i,\tau}-\sum_{i=1}^{n}\eta_{i,\tau_1}\eta_{i,\tau_2}\right\}$$

$$=\sum_{j=1}^{m}\omega_{j,h}(\tau_2)\{\Sigma_\eta(j,\tau_1)-\Sigma_\eta(t,\tau_2)\}+o_p(1), \tag{S.5}$$

where the first two term on the right hand of (S.5) is $O(h^2)$ according to the assumption on the distribution of $\eta_{i,\tau}$; see the equation (26) in the supplementary materials of Zhu et al. (2014).

We next prove the consistency of $n^{-1}\sum_{i=1}^{n}\widehat{\varepsilon}_{i,\tau}^2$. Notice that

$$\widehat{\varepsilon}_{i,\tau}=\widehat{e}_{i,\tau}-\widehat{\eta}_{i,\tau}=y_{i,\tau}-Z_{i,\tau}^\top\widehat{\theta}(\tau)-\widehat{\eta}_{i,\tau}.$$

Similarly to the proof of (a), we denote $\Delta_\theta(\tau)=\widehat{\theta}(\tau)-\theta(\tau)$, and $\Delta_{\varepsilon_i}(\tau)=-Z_{i,\tau}^\top\Delta_\theta(\tau)-\Delta_{\eta_i}(\tau)$. It follows that

$$n^{-1}\sum_{i=1}^{n}\widehat{\varepsilon}_{i,\tau}^2=n^{-1}\sum_{i=1}^{n}\varepsilon_{i,\tau}^2+n^{-1}\sum_{i=1}^{n}\Delta_{\varepsilon_i}^2(\tau)+2n^{-1}\sum_{i=1}^{n}\varepsilon_{i,\tau}\Delta_{\varepsilon_i}^2(\tau).$$

The first term $n^{-1}\sum_{i=1}^{n}\varepsilon_{i,\tau}^2$ converges to $\sigma_\varepsilon^2(\tau)$ according to the Law of Large Number, and the other two terms both converge to zero based on the same arguments used before. We omit the details to save space.

Finally, recall that $\widehat{\boldsymbol{V}}_\theta$ is the sandwich estimator of $\boldsymbol{V}_{\widehat{\theta}}$ defined in (13). It is straightforward to show that $\sup_{\tau_1,\tau_2}|\widehat{\boldsymbol{V}}_\theta(\tau_1,\tau_2)-\boldsymbol{V}_{\widehat{\theta}}(\tau_1,\tau_2)|=o_p(1)$ based on $\sup_{\tau_1,\tau_2}|\widehat{\Sigma}_y(\tau_1,\tau_2)-\Sigma_y(\tau_1,\tau_2)|=o_p(1)$. Similarly, we can derive that $\sup_{\tau_1,\tau_2}|\widetilde{\boldsymbol{V}}(\tau_1,\tau_2)-\boldsymbol{V}_{\widetilde{\theta}}(\tau_1,\tau_2)|=o_p(1)$. We omit the details to save space. □

S.5. Proof of Theorem 1

Proof: Argument (i) in Theorem 1 can be directly proven based on the properties of the ordinary least square estimator. We focus on proving Argument (ii). Notice that $\widetilde{\theta}(\tau)$ can essentially rewritten as a linear combination of $\{\widehat{\theta}(k)\}_k$, i.e.,

$$\widetilde{\theta}(\tau)=\sum_{k=1}^{m}\omega_{k,h}(\tau)\widehat{\theta}(k)=\sum_{k=1}^{m}\omega_{k,h}(\tau)\{\widehat{\theta}(k)-\theta(k)+\theta(k)-\theta(\tau)+\theta(\tau)\}$$

$$=\theta(\tau)+\sum_{k=1}^{m}\omega_{k,h}(\tau)\{\widehat{\theta}(k)-\theta(k)\}+\sum_{k=1}^{m}\omega_{k,h}(\tau)\{\theta(k)-\theta(\tau)\}.$$

It follows that

$$E\{\widetilde{\theta}(\tau)-\theta(\tau)\}$$

$$=\sum_{k=1}^{m}\omega_{k,h}(\tau)\{\widehat{\theta}(k)-\theta(\tau)\}$$

$$=\sum_{k=1}^{m}\omega_{k,h}(\tau)\{\theta(k)-\theta(\tau)\}$$

$$=\left\{\sum_{k=1}^{m}\frac{1}{mh}K\left(\frac{\tau-k}{mh}\right)\right\}^{-1}\cdot\left[\sum_{k=1}^{m}\frac{1}{mh}K\left(\frac{\tau-k}{mh}\right)\{\theta(k)-\theta(\tau)\}\right]$$

Denote

$$f(\tau)=\sum_{k=1}^{m}\frac{1}{mh}K\left(\frac{\tau-k}{mh}\right)$$

$$g_1(\tau)=\sum_{k=1}^{m}\frac{1}{mh}K\left(\frac{\tau-k}{mh}\right)\{\theta(k)-\theta(\tau)\}$$

Note that $f(\tau)\longrightarrow 1$, it suffices to bound $|g_1(\tau)|$. Define

$$g_2(\tau)=\int_0^1\frac{1}{h}K\left(\frac{um-\tau}{mh}\right)\{\theta(um)-\theta(\tau)\}du.$$

By decomposing $g_1(\tau) = g_2(\tau) + \{g_1(\tau) - g_2(\tau)\}$, we first show $g_2(\tau) = O\left(h^2\right)$, and then prove $g_1(\tau) - g_2(\tau) = O\left(m^{-1}\right)$. The time domain of interest is fixed, and the increment of m equals the encryption of grids. Define a function θ_0 such that $\theta_0(\cdot)$ such that $\theta(\tau) = \theta_0\left(\frac{\tau}{m}\right)$ for any τ. It follows that

$$\theta(s) - \theta(t) = \theta_0\left(\frac{s}{m}\right) - \theta_0\left(\frac{t}{m}\right) = \theta_0'\left(\frac{t}{m}\right)\left(\frac{s-t}{m}\right) + \frac{1}{2}\theta_0''\left(\frac{t}{m}\right)\left(\frac{s-t}{m}\right)^2 + O\left(m^{-3}\right).$$

Then we have

$$\begin{aligned}
g_2(\tau) &= \int_0^1 \frac{1}{h}K\left(\frac{u-\tau/m}{h}\right)\left\{\theta_0(u) - \theta_0\left(\frac{\tau}{m}\right)\right\} du \\
&= \int_0^1 \frac{1}{h}K\left(\frac{u-\tau/m}{h}\right)\left\{\theta_0'\left(\frac{\tau}{m}\right)\left(u - \frac{\tau}{m}\right) + \theta_0''\left(\frac{\tau}{m}\right)\left(u - \frac{\tau}{m}\right)^2\right\} du \\
&= \int_0^1 K\left(\frac{u-\tau/m}{h}\right)\cdot\left(\frac{u-\tau/m}{h}\right)^2 \cdot \theta_0''\left(\frac{\tau}{m}\right) h^2 d\left(\frac{u-\tau/m}{h}\right) \\
&= O\left(h^2\right).
\end{aligned}$$

Note that for any second-order continuous function f_0,

$$\int_a^b f_0(x)dx = \frac{1}{2}(b-a)\{f_0(a) + f_0(b)\} - \frac{1}{12}(b-a)^3 f_0''(\xi)$$

for some $\xi \in (a, b)$. Let

$$s(u) = \frac{1}{h}K\left(\frac{u-\tau/m}{h}\right)\left\{\theta_0(u) - \theta_0\left(\frac{\tau}{m}\right)\right\}.$$

Then where exists some $\xi_k \in (k-1, k)$ such that

$$\begin{aligned}
g_2(\tau) &= \sum_{k=1}^m \int_{(k-1)/m}^{k/m} s(u)du \\
&= \sum_{k=1}^m \frac{1}{2m}\{s(k) + s(k-1)\} - \frac{1}{12m}\sum_{k=1}^m s''\left(\xi_k\right) \\
&= g_1(\tau) + \frac{1}{2m}\{s(0) - s(m)\} - \frac{1}{12m}\sum_{k=1}^m s''\left(\xi_k\right)
\end{aligned}$$

Hence

$$g_2(\tau) - g_1(\tau) = \frac{1}{2m}\{s(0) - s(m)\} - \frac{1}{12m}\sum_{k=1}^m s''\left(\xi_k\right).$$

We can represent $(12m)^{-1}\sum_{k=1}^m s''(\xi_k)$ as the summation of the follow three quantities:

$$\frac{1}{12m^3h^3}\sum_{k=1}^m K''\left(\frac{\xi_k - \tau}{mh}\right)\left\{\theta_0\left(\frac{\xi_k}{m}\right) - \theta_0\left(\frac{\tau}{m}\right)\right\} \approx \frac{1}{12m^2h^2}\int_0^1 \frac{1}{h}K''\left(\frac{u-\tau/m}{h}\right)\left\{\theta_0(u) - \theta_0\left(\frac{\tau}{m}\right)\right\} = O(m^{-2}),$$

$$\frac{1}{12m^3h^2}\sum_{k=1}^m K'\left(\frac{\xi_k - \tau}{mh}\right)\theta_0'(\xi_k/m) \approx \frac{1}{12m^2h}\int_0^1 \frac{1}{h}K'\left(\frac{u-\tau/m}{h}\right)\theta_0'(u)du = O(m^{-2}h^{-1}),$$

$$\frac{1}{12m^3h}\sum_{k=1}^m K\left(\frac{\xi_k - \tau}{mh}\right)\theta_0''(\xi_k/m) \approx \frac{1}{12m^2}\int_0^1 K\left(\frac{u-\tau/m}{h}\right)\theta_0''(u)du = O(m^{-2}).$$

It follows that $g_2(\tau) - g_1(\tau) = O(m^{-1})$ and the bias term satisfies

$$g_1(\tau) = O(m^{-1} + h^2). \tag{S.6}$$

As for the covariance matrix, we have that

$$\begin{aligned}
&\mathrm{Cov}\{\tilde{\theta}(\tau), \tilde{\theta}(s)\} \\
&= \mathrm{Cov}\left\{\sum_{k=1}^m w_h(\tau - k)\widehat{\theta}(k), \Sigma_{l=1}^m w_h(s-l)\widehat{\theta}(l)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= E\left[\sum_{k=1}^{m}\sum_{l=1}^{m} w_h(\tau-k)w_h(s-l)\{\widehat{\theta}(k)-\theta(k)\}\{\widehat{\theta}(l)-\theta(l)\}\right] \\
&= \frac{1}{n}\sum_{k=1}^{m}\sum_{l=1}^{m} w_h(\tau-k)w_h(s-l)V_{\widehat{\theta}}(k,l) \\
&= \frac{1}{n}\cdot\frac{\widehat{g}(\tau,s)}{f(\tau)\cdot f(s)},
\end{aligned}
$$

where $V_{\widehat{\theta}}(k,l) = \mathrm{Cov}\{\widehat{\theta}(k),\widehat{\theta}(l)\} \in \mathbb{R}^{p\times p}$ and that

$$
\widehat{g}(\tau,s) = \frac{1}{nm^2h^2}\left[\sum_{k=1}^{m}\sum_{l=1}^{m} K\left(\frac{\tau-k}{mh}\right)K\left(\frac{s-l}{mh}\right)V_{\widehat{\theta}}(k,l)\right].
$$

Let

$$
\begin{aligned}
V_\varepsilon &= V_{\widehat{\theta}} - V_{\widetilde{\theta}} \\
&= \left(EZ_i^\top Z_i\right)^{-1}\cdot E\left(Z_i^\top \Sigma_\varepsilon Z_i^\top\right)\cdot\left(EZ_i^\top Z_i\right)^{-1} \\
&= \mathrm{diag}\left\{\sigma_j^2\left(EZ_{ij}Z_{ij}^\top\right)^{-1}\right\}_{j=1,\dots,m},
\end{aligned}
$$

and $V_\varepsilon(k) = \sigma_k^2\left(EZ_{ik}Z_{ik}^\top\right)^{-1}$. Then we can represent

$$
\widehat{g}(\tau,s) = \widehat{g}_1(\tau,s) + \widehat{g}_2(\tau,s),
$$

where

$$
\begin{aligned}
\widehat{g}_1(\tau,s) &= \frac{1}{nm^2h^2}\left[\sum_{k=1}^{m}\sum_{l=1}^{m} K\left(\frac{\tau-k}{mh}\right)K\left(\frac{s-l}{mh}\right)V_{\widetilde{\theta}}(k,l)\right], \\
\widehat{g}_2(\tau,s) &= \frac{1}{nm^2h^2}\left[\sum_{k=1}^{m} K\left(\frac{\tau-k}{mh}\right)K\left(\frac{s-k}{mh}\right)V_\varepsilon(k)\right].
\end{aligned}
$$

Using the same arguments in (S.6), we have

$$
\begin{aligned}
\widehat{g}_1(\tau,s) &= \frac{1}{n}V_{\widetilde{\theta}}(\tau,s) + O(n^{-1}m^{-1} + n^{-1}h^2), \\
\widehat{g}_2(\tau,s) &= O(n^{-1}m^{-1}).
\end{aligned}
$$

The above arguments implies that for any vector $\boldsymbol{a}_{n,2}$ with unit ℓ_2 norm, the asymptotic bias of $\sqrt{n}\boldsymbol{a}_{n,2}^\top(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta})$ is upper bounded by $n^{-1/2}\|\boldsymbol{a}_{n,2}\|_2\|\mathbb{E}\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_2 = O(\sqrt{n}h^2+\sqrt{n}m^{-1})$, using Cauchy-Schwarz inequality, and that its asymptotic variance is given by $\boldsymbol{a}_{n,2}^\top\boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2}$. Under the assumption that $\lambda_{\min}(\boldsymbol{a}_{n,2}^\top\boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2})$ is bounded away from zero, the bias of $\sqrt{n}\boldsymbol{a}_{n,2}^\top(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta})/\sqrt{\boldsymbol{a}_{n,2}^\top\boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2}}$ is bounded by $O(\sqrt{n}h^2+\sqrt{n}m^{-1})$ as well.

It remains to prove the asymptotic normality of $\sqrt{n}\boldsymbol{a}_{n,2}^\top(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta})$. Let $\boldsymbol{a}_{n,2} = (a_{n,2,1}^\top, a_{n,2,2}^\top,\cdots,a_{n,2,m}^\top)^\top$ where each $a_{n,2,\tau}$ corresponds to a $(d+2)$-dimensional vector. The key observation is that, $\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}$ is a linear transformation of $\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}$, which is equivalent to a sum of independent random vectors, given by

$$
n^{-1/2}\sum_{i=1}^{n}\sum_{\tau=1}^{m}\sum_{k=1}^{m}\omega_{k,h}(\tau)a_{n,2,\tau}^\top(\mathbb{E}Z_{i,k}Z_{i,k}^\top)^{-1}Z_{i,k}\eta_{i,k} + o_p(1).
$$

We aim to apply Lindeberg central limit theorem to show the asymptotic normality. It remains to verify the Lindeberg condition:

$$
(\boldsymbol{a}_{n,2}^\top\boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2})^{-1}\mathbb{E}|\sum_{\tau=1}^{m}\sum_{k=1}^{m}\omega_{k,h}(\tau)a_{n,2,\tau}^\top(\mathbb{E}Z_{i,k}Z_{i,k}^\top)^{-1}Z_{i,k}\eta_{i,k}|^2
$$

$$\times \mathbb{I}(|\sum_{\tau=1}^{m}\sum_{k=1}^{m}\omega_{k,h}(\tau)a_{n,2,\tau}^\top(\mathbb{E}Z_{i,k}Z_{i,k}^\top)^{-1}Z_{i,k}\eta_{i,k}| > \epsilon\sqrt{n\boldsymbol{a}_{n,2}^\top \boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2}}) \to 0,$$

for any $\epsilon > 0$. The left-hand-side is uniformly bounded by 1. As such, it suffices to show

$$\mathbb{P}\left(|\sum_{\tau=1}^{m}\sum_{k=1}^{m}\omega_{k,h}(\tau)a_{n,2,\tau}^\top(\mathbb{E}Z_{i,k}Z_{i,k}^\top)^{-1}Z_{i,k}\eta_{i,k}| > \epsilon\sqrt{n\boldsymbol{a}_{n,2}^\top \boldsymbol{V}_{\widetilde{\theta}}\boldsymbol{a}_{n,2}}\right) \to 0.$$

However, this follows directly by the Chebyshev's inequality.

Finally, it is proven in Lemma 1 that $\widetilde{\boldsymbol{V}}_{\theta}$ is a consistent estimate of $\boldsymbol{V}_{\widetilde{\theta}}$. As such, $\widehat{se}(\widehat{\mathrm{DE}})$ is a consistent estimate of $se(\widehat{\mathrm{DE}})$. Argument (iii) thus follows. □

S.6. Proof of Theorem 2

We focus on provide an upper error bound for

$$\rho^*(z) = \left|\mathbb{P}\left(\frac{1}{m}\widehat{\mathrm{IE}} - \frac{1}{m}\mathrm{IE} \le z\right) - \mathbb{P}\left(\frac{1}{m}\widehat{\mathrm{IE}}^b - \frac{1}{m}\widehat{\mathrm{IE}} \le z\middle|\mathrm{Data}\right)\right|.$$

We begin with some notations. Note that $\widetilde{\theta}(\tau)$ can be expressed as

$$\widetilde{\theta}(\tau) = \theta_s(\tau) + \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{m} B_{i,k}(\tau)e_{i,k}\right),$$

where

$$B_{i,k}(\tau) = \omega_{k,h}(\tau)\left(\frac{1}{n}\sum_{i'=1}^{n} Z_{i',k}^\top Z_{i',k}\right)^{-1} Z_{i,k}$$

are independent of the random part e_i, and $\theta_s(\tau) = \sum_k \omega_{k,h}(\tau)\theta(k)$. Let $e_{i,\tau}^{\theta} = \sum_{k=1}^{m} B_{i,k}(\tau)e_{i,k} = \{e_{i,\tau}^{\beta_0}, (e_{i,\tau}^{\beta})^\top, e_{i,\tau}^{\gamma}\}^\top$ and $e_{\tau}^{\theta} = n^{-1/2}\sum_{i=1}^{n} e_{i,\tau}^{\theta}$. Similarly, we can represent $\widetilde{\Theta}(\tau)$ as

$$\widetilde{\Theta}(\tau) = \Theta_s(\tau) + \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{m-1} B_{i,k}(\tau)E_{i,k}\right),$$

where $\Theta_s(\tau) = \sum_k \omega_{k,h}(\tau)\Theta(k)$. Let $E_{i,\tau}^{\Theta} = \sum_{k=1}^{m} B_{i,k}(\tau)E_{i,k} = \{E_{i,\tau}^{\phi_0}, (E_{i,\tau}^{\Phi})^\top, E_{i,\tau}^{\Gamma}\}^\top$ and $E_{\tau}^{\Theta} = n^{-1/2}\sum_{i=1}^{n} E_{i,\tau}^{\Theta}$. It follows that

$$\widetilde{\beta}(\tau) = \beta_s(\tau) + \frac{1}{\sqrt{n}}e_{\tau}^{\beta},\ \widetilde{\Phi}(\tau) = \Phi_s(\tau) + \frac{1}{\sqrt{n}}E_{\tau}^{\Phi},\ \widetilde{\Gamma}(\tau) = \Gamma_s(\tau) + \frac{1}{\sqrt{n}}E_{\tau}^{\Gamma}.$$

The OLS estimation corresponds to the special case $h = 0$. We remark that E_{τ}^{Θ} is asymptotically normal when $h = 0$ and degenerates to a point distribution when $mh \to \infty$. To make the following analysis hold for the OLS-based test statistic, we view E_{τ}^{Θ} as a random variable in the discussion below.

For simplicity, let $\mathrm{vec}(\cdot)$ be the operator that reshapes a matrix into a vector by stacking its columns on top of one another. Denote

$$\begin{aligned} x_{i,\tau} &= \left[(e_{i,\tau}^{\beta})^\top, \{\mathrm{vec}(E_{i,\tau}^{\Phi})\}^\top, (E_{i,\tau}^{\Gamma})^\top\right]^\top \in \mathbb{R}^{2d(d+2)}, \\ x_i &= \left(x_{i,2}^\top, x_{i,3}^\top, \ldots, x_{i,m}^\top\right)^\top \in \mathbb{R}^{p_x}, \quad p_x = 2(m-1)dp,\ d = p-2. \end{aligned} \tag{S.7}$$

Let $\{y_i\}_i$ be independent mean zero Gaussian vectors with $\mathbb{E}y_iy_i^\top = \mathbb{E}x_ix_i^\top$. We similarly represent y_i as

$$\begin{aligned} y_{i,\tau} &= \left[(\overline{e}_{i,\tau}^{\beta})^\top, \{\mathrm{vec}(\overline{E}_{i,\tau}^{\Phi})\}^\top, (\overline{E}_{i,\tau}^{\Gamma})^\top\right]^\top \in \mathbb{R}^{2d(d+2)}, \\ y_i &= \left(y_{i,2}^\top, y_{i,3}^\top, \ldots, y_{i,m}^\top\right)^\top \in \mathbb{R}^{p_x}. \end{aligned} \tag{S.8}$$

Let $\left\{e_{i,j}^b, E_{i,j}^b\right\}$ be the empirical Gaussian analogs of $\{e_{i,j}, E_{i,j}\}$. In other words, for $i = 1, \dots, n$, $j = 1, \dots, m$, let

$$e_{i,j}^b = \widehat{e}_{i,j}\xi_i,\ E_{i,j}^b = \widehat{E}_{i,j}\xi_i,$$

where $\xi_1, \dots, \xi_n$ are i.i.d standard normal random variables. We next define

$$\begin{aligned} w_{i,\tau} &= \left[(e_{i,\tau}^{\beta,b})^\top, \{\mathrm{vec}(E_{i,\tau}^{\Phi,b})\}^\top, (E_{i,\tau}^{\Gamma,b})^\top\right]^\top \in \mathbb{R}^{2d(d+2)}, \\ w_i &= \left(w_{i,2}^\top, w_{i,3}^\top, \dots, w_{i,m}^\top\right)^\top \in \mathbb{R}^{p_x}. \end{aligned} \tag{S.9}$$

Let

$$\begin{aligned} X &= (X_2^\top, X_3^\top, \dots, X_m^\top) = \frac{1}{\sqrt{n}}\sum_{i=1}^n x_i, \\ Y &= (Y_2^\top, Y_3^\top, \dots, Y_m^\top) = \frac{1}{\sqrt{n}}\sum_{i=1}^n y_i, \\ W &= (W_2^\top, W_3^\top, \dots, W_m^\top) = \frac{1}{\sqrt{n}}\sum_{i=1}^n w_i. \end{aligned}$$

Define the following function

$$F_{\mathrm{IE}}(X;\theta,\Theta) \equiv \frac{1}{m}\sum_{l=2}^m \left[\left(\beta(l) + \frac{e_l^\beta}{\sqrt{n}}\right)^\top \sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}\left(\Phi(k) + \frac{E_k^\Phi}{\sqrt{n}}\right)\left(\Gamma(j) + \frac{E_j^\Gamma}{\sqrt{n}}\right)\right\}\right].$$

We next represent the proposed test statistic and the bootstrap samples based on F_{IE}. Recall that $\Theta_s(\tau) = \sum_k \omega_{k,h}(\tau)\Theta(k)$ and $\theta_s(\tau) = \sum_k \omega_{k,h}(\tau)\theta(k)$ are the smoothed parameters, and $\widetilde{\theta}, \widetilde{\Theta}$ correspond to the estimates. The difference between the proposed test statistic and the oracle indirect effect $m^{-1}(\widehat{\mathrm{IE}} - \mathrm{IE})$ can be represented as $T_0^* = F_{\mathrm{IE}}(X;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta,\Theta)$. Similarly, we can represent $m^{-1}(\widehat{\mathrm{IE}}^b - \widehat{\mathrm{IE}})$ by $W_0^* = F_{\mathrm{IE}}(W;\widetilde{\theta},\widetilde{\Theta}) - F_{\mathrm{IE}}(0;\widetilde{\theta},\widetilde{\Theta})$. By definition, we have

$$\rho^*(z) = \Big| P\left\{T_0^* \le z\right\} - P\left\{W_0^* \le z\right\}\Big|. \tag{S.10}$$

We also define the oracle statistics: $T_0 = F_{\mathrm{IE}}(X;\theta,\Theta) - F_{\mathrm{IE}}(0;\theta,\Theta) = F_{\mathrm{IE}}(X) - F_{\mathrm{IE}}(0)$, $Z_0 = F_{\mathrm{IE}}(Y;\theta,\Theta) - F_{\mathrm{IE}}(0;\theta,\Theta) = F_{\mathrm{IE}}(Y) - F_{\mathrm{IE}}(0)$, $W_0 = F_{\mathrm{IE}}(W;\theta,\Theta) - F_{\mathrm{IE}}(0;\theta,\Theta) = F_{\mathrm{IE}}(Z) - F_{\mathrm{IE}}(0)$ by replacing θ_s, $\widetilde{\theta}$, Θ_s and $\widetilde{\Theta}$ with the oracle values. This yields an upper bound for

$$\rho(z) = \Big| P\left\{T_0 \le z\right\} - P\left\{W_0 \le z\right\}\Big|. \tag{S.11}$$

The proof is divided into two parts. We first provide an upper error bound for $\sup_z \rho(z)$, showing that T_0 can be well-approximated by W_0. See Lemma S.3 below. Then, we provide upper error bounds for the difference between W_0 and W_0^*, and the difference between T_0 and T_0^*. This yields the error bound for $\sup_z \rho^*(z)$.

LEMMA S.3. *Under the conditions of Theorem 2,* $\sup_z \rho(z) \le Cn^{-1/8}$ *for some constant* $C > 0$.

We first outline the main idea of the proof. We then present the details. The proof is based on the high-dimensional Gaussian approximation theory developed by Chernozhukov et al. (2013). In their paper, they developed a coupling inequality for maxima of sums of high-dimensional random vectors. They began by approximating the maximum function using a smooth surrogate and then developed a coupling inequality for the smooth function of the high-dimensional random vector.

In our setup, the statistic T_0 can be represented as a smooth function of sums of random vectors whose dimension is allowed to diverge with the sample size. Such an observation allows us to employ the coupling inequality to establish the size and power property of the proposed test. The proof of Lemma S.3 contains two main parts. In the first part, we assume the covariance of the time-varying covariates is known and employ Slepian interpolation, Stein's leave-one-out method as well as a truncation method to bound the Kolmogorov distance between the distributions of T_0 and its Gaussian analog Z_0. In the second part, we establish the validity of the multiplier bootstrap for estimating quantiles of Z_0 when the covariance matrix is unknown, i.e., W_0. The detailed proof is given as follows.

Proof of Lemma S.3: Define function $g(s) = g_0\big(\psi(s-t)\big)$ for some constant $\psi > 0$ and some thrice differentiable function g_0 that satisfies $g_0(s) = 1$ when $s \le 0$, $g_0(s) = 0$ when $s \ge 1$ and $g_0(s) \ge 0$ otherwise. Let $m = g \circ F_{\text{IE}}$. We also introduce the following notations: $\mathbb{E}_n(\cdot) = n^{-1}\sum_{i=1}^n(\cdot)$; $\overline{\mathbb{E}}(\cdot) = \mathbb{E}_n\mathbb{E}(\cdot)$; C^k denotes the class of k times continuously differentiable functions; C_b^k denotes the class of functions $f \in C^k$ and $\sum_z |\partial^j f(z)/\partial z^j|$ for $j = 0, \ldots, k$; $a \lesssim b$ if a is smaller than or equal to b up to a universal positive constant; $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. We define the Slepian interpolation $Z(t)$ between Y and X, Stein's leave-one-out version $Z^{(i)}(t)$ of $Z(t)$, and other useful terms as follows:

$$Z(t) = \sqrt{t}X + \sqrt{1-t}Y = \sum_{i=1}^n Z_i(t), \quad z_i(t) = n^{-1/2}(\sqrt{t}x_i + \sqrt{1-t}y_i),$$
$$Z^{(i)}(t) = Z(t) - Z_i(t), \quad \dot{z}_{ij}(t) = \frac{1}{2\sqrt{n}}\left(\frac{1}{\sqrt{t}}x_{ij} - \frac{1}{\sqrt{1-t}}y_{ij}\right).$$

We first prove

$$\sup_{t\in\mathbb{R}} |P(T_0 \le t) - P(Z_0 \le t)| \le C' n^{-1/8}, \tag{S.12}$$

where $C' > 0$ is a constant. From the construction of $g(\cdot)$, we have $G_k \lesssim \psi^k, k = 0, 1, 2, 3$ where $G_k = \sup_{z\in\mathbb{R}} |\partial^k g(z)|, k \ge 0$, and

$$\begin{aligned}
&P(T_0 \le t) = P\big(F_{\text{IE}}(X) \le t\big) \le \mathbb{E}g\big(F_{\text{IE}}(X)\big),\\
&\mathbb{E}g\big(F_{\text{IE}}(Y)\big) \le P\big(F_{\text{IE}}(Y) \le t + \psi^{-1}\big),\\
&P(Z_0 \le t + \psi^{-1}) = P\big(F_{\text{IE}}(Y) \le t + \psi^{-1}\big) \ge \mathbb{E}g\big(F_{\text{IE}}(Y)\big),
\end{aligned}$$

which give the decompose

$$P(T_0 \le t) - P(Z_0 \le t) \le \underbrace{\big\{\mathbb{E}g\big(F_{\text{IE}}(X)\big) - \mathbb{E}g\big(F_{\text{IE}}(Y)\big)\big\}}_{(a)} + \underbrace{\big\{P(Z_0 \le t + \psi^{-1}) - P(Z_0 \le t)\big\}}_{(b)}.$$

In the following, we calculate (a) in Steps 1-2 and derive the bound for (b) in Step 3.

Step 1. We first calculate the upper bounds of (a). We have by Taylor's expansion,

$$\mathbb{E}\{m(X) - m(Y)\} = \sum_{j=1}^{p_x}\sum_{i=1}^n \int_0^1 \mathbb{E}\{\partial_j m(Z(t))\dot{Z}_{ij}(t)\}dt = I + II + III,$$

where

$$\begin{aligned}
I &= \sum_{j=1}^{p_x}\sum_{i=1}^n \int_0^1 \mathbb{E}\{\partial_j m(Z^{(i)}(t))\dot{Z}_{ij}(t)\}dt,\\
II &= \sum_{j,k=1}^{p_x}\sum_{i=1}^n \int_0^1 \mathbb{E}\{\partial_j\partial_k m(Z^{(i)}(t))\dot{Z}_{ij}(t)Z_{ik}(t)\}dt,\\
III &= \sum_{j,k,l=1}^{p_x}\sum_{i=1}^n \int_0^1\int_0^1 (1-s)\mathbb{E}\{\partial_j\partial_k\partial_l m(Z^{(i)}(t) + sZ_{i,t})\dot{Z}_{ij}(t)Z_{ik}(t)Z_{il}(t)\}dsdt.
\end{aligned}$$

By independence of $Z^{(i)}(t)$ and $\dot{Z}_{ij}(t)$ together with the fact that $\mathbb{E}\{\dot{Z}_{ij}(t)\} = 0$, we have $I = 0$. Note that $Z^{(i)}(t)$ is independent of $\dot{Z}(t)Z_{ik}(t)$, and $\mathbb{E}\{\dot{Z}_{ij}(t)Z_{ik}(t)\} = n^{-1}\mathbb{E}\{x_{ij}x_{ik} - y_{ij}y_{ik}\}$,

$$II = \sum_{j,k=1}^{p_x}\sum_{i=1}^n \int_0^1 \mathbb{E}\{\partial_j\partial_k m(Z^{(i)}(t))\}\mathbb{E}\{\dot{Z}_{ij}(t)Z_{ik}(t)\}dt = 0.$$

We now prove $(a) \le |III| \lesssim \psi^3 n^{-2} + \psi^2 n^{-2} + \psi n^{-2}$ in Step 2.

Step 2. Note that

$$III = \sum_{j,k,l=1}^{p_x}\sum_{i=1}^n \int_0^1 \left[\mathbb{E}\left\{\int_0^1 \partial_j\partial_k\partial_l m(Z^{(i)}(t) + sZ_i(t))ds\right\}\dot{Z}_{ij}(t)Z_{ik}(t)Z_{il}(t)\right]dt$$

$$\simeq \sum_{j,k,l=1}^{p_x} \sum_{i=1}^{n} \int_0^1 \mathbb{E}\partial_j\partial_k\partial_l m(Z(t))\dot{Z}_{ij}(t)Z_{ik}(t)Z_{il}(t)dt,$$

where

$$\partial_j\partial_k\partial_l m(Z) \simeq \psi^3\partial_j F_{\mathrm{IE}}(Z)\partial_k F_{\mathrm{IE}}(Z)\partial_l F_{\mathrm{IE}}(Z) + \psi^2\partial_j F_{\mathrm{IE}}(Z)\partial_k\partial_l F_{\mathrm{IE}}(Z) + \psi\partial_j\partial_k\partial_l F_{\mathrm{IE}}(Z).$$

Note that

$$\begin{aligned} |III| &\le \sum_{j,k,l=1}^{p_x} \sum_{i=1}^{n} \int_0^1 \sqrt{\mathbb{E}|\partial_j\partial_k\partial_l m(Z(t))|^2}\sqrt{\mathbb{E}|\dot{Z}_{ij}(t)Z_{ik}(t)Z_{il}(t)|^2}dt \\ &\le \int_0^1 \left(\sum_{j,k,l=1}^{p_x} \sqrt{\mathbb{E}|\partial_j\partial_k\partial_l m(Z(t))|^2}\right)\left(\max_{1\le j,k,l\le p_x} n\overline{\mathbb{E}}|\dot{Z}_{ij}(t)Z_{ik}(t)Z_{il}(t)|\right)dt. \end{aligned} \quad \text{(S.13)}$$

We first compute $\sum_{j,k,l=1}^{p_x}\sqrt{\mathbb{E}|\partial_j\partial_k\partial_l m(Z(t))|^2}$. Define function

$$\mathcal{G} = \mathbb{1}\left\{\max_{1\le j\le p_x/2}\left|u_j/\sqrt{n}\right| < (1-q)/2\right\},$$

where

$$\begin{aligned} u &= \left((e_2^{\beta})^\top, \{\mathrm{vec}(E_2^{\Phi})\}^\top, (E_2^{\Gamma})^\top, \cdots, (e_m^{\beta})^\top, \{\mathrm{vec}(E_m^{\Phi})\}^\top, (E_m^{\Gamma})^\top\right)^\top \\ &= (u_1, u_2, \cdots, u_{p_x/2})^\top. \end{aligned}$$

Then we have

$$\begin{aligned} \sqrt{\mathbb{E}\{\partial_j\partial_k\partial_l m(Z)\}^2} =& \sqrt{\mathbb{E}\{\partial_j\partial_k\partial_l m(Z)\}^2\mathcal{G} + \mathbb{E}\{\partial_j\partial_k\partial_l m(Z)\}^2\{1-\mathcal{G}\}} \\ \simeq& \psi^3\mathbb{E}(\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}\mathcal{G}) + \psi^3\mathbb{E}\{\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \\ &+ \psi^2\mathbb{E}(\partial_j\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}\mathcal{G}) + \psi^2\mathbb{E}\{\partial_j\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \\ &+ \psi\mathbb{E}(\partial_j\partial_k\partial_l F_{\mathrm{IE}}\mathcal{G}) + \psi\mathbb{E}\{\partial_j\partial_k\partial_l F_{\mathrm{IE}}(1-\mathcal{G})\}. \end{aligned}$$

In the following, we focus on establishing the upper error bounds for $\sum_{j,k,l}\mathbb{E}(\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}\mathcal{G})$ and $\sum_{j,k,l}\mathbb{E}\{\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}(1-\mathcal{G})\}$. The other bounds can be derived similarly.

2.1 *The bound of* $\sum_{j,k,l}\mathbb{E}(\partial_j F_{IE}\partial_k F_{IE}\partial_l F_{IE}\mathcal{G})$.
Let $\overline{q} = (1+q)/2$. Notice that

$$\sum_{j,k,l}\mathbb{E}(\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}\mathcal{G}) \lesssim m^3\mathbb{E}|\partial_j F_{\mathrm{IE}}\mathcal{G}|^3.$$

We next compute $\mathbb{E}|\partial_j F_{\mathrm{IE}}\mathcal{G}|$, which belongs to either one of the following three categories:

$$\begin{aligned} \left|\frac{\partial F_{\mathrm{IE}}}{\partial e_\tau^{\beta}}\mathcal{G}\right| &= m^{-1}n^{-1/2}\left|\sum_{j=1}^{t-1}\left\{\prod_{k=j+1}^{t-1}\left(\Phi(k) + \frac{E_k^{\Phi}}{\sqrt{n}}\right)\left(\Gamma(j) + \frac{E_j^{\Gamma}}{\sqrt{n}}\right)\right\}\mathcal{G}\right| \\ &\lesssim m^{-1}n^{-1/2}\sum_{j=1}^{t-1}\overline{q}^{t-j-1}\{M_\Gamma + (1-q)/2\} \\ &\simeq m^{-1}n^{-1/2}; \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial F_{\mathrm{IE}}}{\partial E_j^{\Gamma}}\mathcal{G}\right| &= m^{-1}n^{-1/2}\left|\sum_{t=j+1}^{m}\left(\beta(\tau) + \frac{e_\tau^{\beta}}{\sqrt{n}}\right)^\top\prod_{k=j+1}^{t-1}\left(\Phi(k) + \frac{E_k^{\Phi}}{\sqrt{n}}\right)\mathcal{G}\right| \\ &\le m^{-1}n^{-1/2}\{M_\beta + (1-q)/2\}\sum_{t=j+1}^{m}\overline{q}^{t-1-j} \end{aligned}$$

$$\simeq m^{-1}n^{-1/2};$$

$$\begin{aligned}\left|\frac{\partial F_{\mathrm{IE}}}{\partial E_l^{\Phi}}\mathcal{G}\right| &= m^{-1}n^{-1/2}\left|\sum_{t=2}^{m}\left(\beta(\tau)+\frac{e_\tau^\beta}{\sqrt{n}}\right)^\top \right.\\ &\qquad \left.\cdot\sum_{j=1}^{t-1}\left\{\prod_{\substack{k\neq l\\ k=j+1}}^{t-1}\left(\Phi(k)+\frac{E_k^\Phi}{\sqrt{n}}\right)\left(\Gamma(j)+\frac{E_j^\Gamma}{\sqrt{n}}\right)\right\}\mathcal{G}\right|\\ &\lesssim m^{-1}n^{-1/2}\sum_{t=l+1}^{m}\sum_{j=1}^{l-1}\overline{q}^{t-2-j}\{M_\beta+(1-q)/2\}\\ &\simeq m^{-1}n^{-1/2}.\end{aligned}$$

It follows that $\sum_{j,k,l}\mathbb{E}(\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}\mathcal{G})\lesssim n^{-3/2}$.

2.2 *The bound of* $\sum_{j,k,l}\mathbb{E}\{\partial_j F_{IE}\partial_k F_{IE}\partial_l F_{IE}(1-\mathcal{G})\}$.
Similarly, we have

$$\sum_{j,k,l}\mathbb{E}\{\partial_j F_{\mathrm{IE}}\partial_k F_{\mathrm{IE}}\partial_l F_{\mathrm{IE}}(1-\mathcal{G})\}\lesssim m^3\mathbb{E}|\partial_j F_{\mathrm{IE}}(1-\mathcal{G})|^3.$$

We consider the derivative with respect to η_τ^β as an example. Notice that

$$\begin{aligned}\mathbb{E}\left\{\frac{\partial F_{\mathrm{IE}}}{\partial\eta_\tau^\beta}(1-\mathcal{G})\right\} &= \mathbb{E}\left|\sum_{j=1}^{t-1}\left\{\prod_{k=j+1}^{t-1}\left(\Phi(k)+\frac{E_k^\Phi}{\sqrt{n}}\right)\left(\Gamma(j)+\frac{E_j^\Gamma}{\sqrt{n}}\right)\right\}(1-\mathcal{G})\right|\\ &\lesssim m^{-1}n^{-1/2}\left[\mathbb{E}\left|\sum_{j=1}^{t-1}\left\{\prod_{k=j+1}^{t-1}\left(\Phi(k)+\frac{E_k^\Phi}{\sqrt{n}}\right)\left(\Gamma(j)+\frac{E_j^\Gamma}{\sqrt{n}}\right)\right\}\right|^2\right]^{1/2}\\ &\qquad\cdot P\left\{\max_{1\le j\le p_x/2}\left|u_j/\sqrt{n}\right|\ge(1-q)/2\right\}.\end{aligned}$$

By Lemma 2.2.10 in Van and Wellner (1996), we have $\mathbb{E}|\max_j u_j|\lesssim\log m$. It follows that

$$\begin{aligned}&\left[\mathbb{E}\left|\sum_{j=1}^{t-1}\left\{\prod_{k=j+1}^{t-1}\left(\Phi(k)+\frac{E_k^\Phi}{\sqrt{n}}\right)\left(\Gamma(j)+\frac{E_j^\Gamma}{\sqrt{n}}\right)\right\}\right|^2\right]^{1/2}\\ \le&\left[\sum_{j=1}^{t-1}\prod_{k=j+1}^{t-1}\mathbb{E}\left|\Phi(k)+\frac{\max_j u_j}{\sqrt{n}}\right|^2\cdot\mathbb{E}\left|\Gamma(j)+\frac{\max_j u_j}{\sqrt{n}}\right|^2\right]^{1/2}\\ \lesssim&\left[\sum_{j=1}^{t-1}\left(1+\frac{\log m}{\sqrt{n}}\right)^{2j}\right]^{1/2}\\ \simeq&\left(1+\frac{\sqrt{n}}{\log m}\right)\left(1+\frac{\log m}{\sqrt{n}}\right)^m\\ \simeq& n^{1/2}(\log m)^{-1}\exp(n^{-1/2}m\log m).\end{aligned}$$

Let $t_0=n^{1/2}(1-q)/2$ and $t_1=t_0-\mathbb{E}\max_j u_j$. Notice that

$$\begin{aligned}P\{\max_j|u_j|>t_0\}=&P\big(\{\max_j u_j>t_0\}\cap\{\max_j|u_j|=\max_j u_j\}\big)\\ &+P\big(\{\min_j u_j<-t_0\}\cap\{\max_j|u_j|=-\min_j u_j\}\big)\\ \le&2P\{\max_j u_j>t_0\}\end{aligned}$$

$$\lesssim P\{|\max_j u_j - \mathbb{E}\max_j u_j| > t_1\}.$$

By Borell TIS inequality and Lemma 2.2.10 in Van and Wellner (1996), we have

$$P\{\max_j |u_j| > t_0\} \lesssim \exp(-t_1^2) \simeq \exp\{-n + 2n^{1/2}\log m - (\log m)^2\}. \tag{S.14}$$

Hence

$$\sum_{j,k,l} \mathbb{E}\{\partial_j F_{\mathrm{IE}} \partial_k F_{\mathrm{IE}} \partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \lesssim n^{-3/2}\delta^3,$$

where

$$\delta = n^{1/2}(\log m)^{-1}\exp\{-n + 2n^{1/2}\log m - (\log m)^2 + n^{-1/2} m \log m\}. \tag{S.15}$$

Combine the above arguments, we obtain

$$\sum_{j,k,l} \mathbb{E}(\partial_j F_{\mathrm{IE}} \partial_k F_{\mathrm{IE}} \partial_l F_{\mathrm{IE}}\mathcal{G}) + \sum_{j,k,l} \mathbb{E}\{\partial_j F_{\mathrm{IE}} \partial_k F_{\mathrm{IE}} \partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \lesssim n^{-3/2}(1+\delta^3),$$

Using similar arguments, we can show that

$$\sum_{j,k,l} \mathbb{E}(\partial_j \partial_k F_{\mathrm{IE}} \partial_l F_{\mathrm{IE}}\mathcal{G}) + \sum_{j,k,l} \mathbb{E}\{\partial_j \partial_k F_{\mathrm{IE}} \partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \lesssim n^{-3/2}(1+\delta^2),$$
$$\sum_{j,k,l} \mathbb{E}(\partial_j \partial_k \partial_l F_{\mathrm{IE}}\mathcal{G}) + \sum_{j,k,l} \mathbb{E}\{\partial_j \partial_k \partial_l F_{\mathrm{IE}}(1-\mathcal{G})\} \lesssim n^{-3/2}(1+\delta).$$

It follows that

$$\sum_{j,k,l=1}^{p_x} \sqrt{\mathbb{E}|\partial_j\partial_k\partial_l m(Z(t))|^2} \lesssim \psi^3 n^{-3/2}(1+\delta^3) + \psi^2 n^{-3/2}(1+\delta^2) + \psi n^{-3/2}(1+\delta), \tag{S.16}$$

where δ depends on m, n through (S.15).

Let $\omega(t) = 1/\min\{\sqrt{t}, \sqrt{1-t}\}$. We observe that

$$\begin{aligned}
&\int_0^1 \max_{j,k,l} n\overline{\mathbb{E}}|\dot{Z}_{ij}(t) Z_{ik}(t) Z_{il}(t)|dt \\
&= \int_0^1 \omega(t) \max_{j,k,l} n\overline{\mathbb{E}}|\{\dot{Z}_{ij}/\omega(t)\}(t) Z_{ik}(t) Z_{il}(t)|dt \\
&\leq_{①} n\int_0^1 \omega(t) \max_{j,k,l}\left(\overline{\mathbb{E}}|\dot{Z}_{ij}/\omega(t)|^3(t)\overline{\mathbb{E}}|Z_{ik}(t)|^3\overline{\mathbb{E}}|Z_{il}(t)|^3\right)^{1/3} dt \\
&\leq_{②} n^{-1/2}\max_j \overline{\mathbb{E}}(|x_{ij}| + |y_{ij}|)^3 \int_0^t \omega(t)dt \\
&\lesssim n^{-1/2}\max_j \overline{\mathbb{E}}|x_{ij}|^3,
\end{aligned} \tag{S.17}$$

where ① is by Hölder inequality and ② follows from the fact that $|\dot{Z}_{ij}/\omega(t)| \leq n^{-1/2}(|x_{ij}| + |y_{ij}|)$, $|Z_{ik}(t)| \leq n^{-1/2}(|x_{ik}| + |y_{ik}|)$.

The condition $m = O(n^{c_2})$ for some $c_2 < 3/2$ implies that $\delta = o(1)$. This together with (S.13), (S.16) and (S.17) yields that

$$(a) = |III| \lesssim \psi^3 n^{-2} + \psi^2 n^{-2} + \psi n^{-2}. \tag{S.18}$$

Step 3. We now derive the upper bound of $(b) \equiv P(Z_0 \leq t + \psi^{-1}) - P(Z_0 \leq t)$. Let $t' = t + F_{\mathrm{IE}}(0)$. Recall that $\overline{e}_\tau^\beta$ is defined in (S.8). Denote $\overline{1} = (1, \ldots, 1)^\top \in \mathbb{R}^d$. Using similar arguments in Step 2.2, we have

$$\begin{aligned}
P(Z_0 \leq t) &\leq P(Z_0\mathcal{G} \leq t) + \mathbb{E}(1-\mathcal{G}) \\
&\lesssim P\left(\frac{1}{m}\sum_{t=2}^m \left(\beta(\tau) + \frac{\overline{e}_\tau^\beta}{\sqrt{n}}\right)^\top \overline{1} \leq t'\right) + \exp\{-n + 2n^{1/2}\log m - (\log m)^2\}
\end{aligned}$$

$$\simeq P\left(\frac{1}{m}\sum_{t=2}^{m}\left(\beta(\tau)+\frac{\overline{e}_\tau^\beta}{\sqrt{n}}\right)^\top \bar{1}\le t'\right), \tag{S.19}$$

where the second inequality is due to the conclusion (S.14) and the third inequality follows from the condition $m = O(n^{c_2})$ for some $c_2 < 3/2$. Notice that $\overline{e}_\tau^\beta$ is a Gaussian random vector, we have

$$\sup|P(Z_0 \le t+\psi^{-1}) - P(Z_0 \le t)| \simeq n^{1/2}\psi^{-1}.$$

To summarize, we have shown that

$$P(T_0 \le t) - P(Z_0 \le t) \lesssim \psi^3 n^{-2} + \psi^2 n^{-2} + \psi n^{-2} + n^{1/2}\psi^{-1}.$$

Take $\psi \simeq n^{5/8}$, we have

$$P(T_0 \le t) - P(Z_0 \le t) \lesssim n^{-1/8}.$$

By Lemma 3.2 of Chernozhukov et al. (2013), we have shown that for $\alpha \in (0,1)$ and $\vartheta > 0$,

$$P(c_{W_0}(\alpha) \le c_{Z_0}(\alpha+\vartheta^{1/2})) \ge 1 - P(\Delta > \vartheta),$$
$$P(c_{Z_0}(\alpha) \le c_{W_0}(\alpha+\vartheta^{1/2})) \ge 1 - P(\Delta > \vartheta),$$

where $c_{W_0}(\alpha)$ and $c_{z_0}(\alpha)$ denote the critical values of W_0 and Z_0 under the significance level α, respectively. Define

$$\rho_\ominus = \sup_{\alpha\in(0,1)} P\Big(\{c_{Z_0}(\alpha) < T_0 \le c_{W_0}(\alpha)\} \cup \{c_{W_0}(\alpha) < T_0 \le c_{Z_0}(\alpha)\}\Big).$$

Note that

$$\begin{aligned}
&P\Big(c_{Z_0}(\alpha) < T_0 \le c_{W_0}(\alpha)\Big)\\
=&P\Big(c_{Z_0}(\alpha) < T_0 \le c_{Z_0}(\alpha+\vartheta^{1/2})\Big) + P\Big(\{c_{Z_0}(\alpha+\vartheta^{1/2}) < T_0 \le c_{W_0}(\alpha)\} \cap \{c_{W_0}(\alpha) > c_{Z_0}(\alpha+\vartheta^{1/2})\}\Big)\\
&- P\Big(\{c_{W_0}(\alpha) < T_0 \le c_{Z_0}(\alpha+\vartheta^{1/2})\} \cap \{c_{W_0}(\alpha) \le c_{Z_0}(\alpha+\vartheta^{1/2})\}\Big)\\
\le&P\Big(c_{Z_0}(\alpha) < T_0 \le c_{Z_0}(\alpha+\vartheta^{1/2})\Big) + P\Big(c_{W_0}(\alpha) > c_{Z_0}(\alpha+\vartheta^{1/2})\Big)\\
\le&P\Big(c_{Z_0}(\alpha) < Z_0 \le c_{Z_0}(\alpha+\vartheta^{1/2})\Big) + \rho + P(\Delta > \vartheta)\\
\le&\vartheta^{1/2} + \rho + P(\Delta > \vartheta).
\end{aligned}$$

Similarly, we can show

$$P\Big(c_{W_0}(\alpha) < T_0 \le c_{Z_0}(\alpha)\Big) \le \vartheta^{1/2} + \rho + P(\Delta > \vartheta).$$

By the definition of $\rho_\ominus$, we have

$$\rho_\ominus \le 2\vartheta^{1/2} + 2P(\Delta > \vartheta) + 2\rho.$$

On the other hand,

$$\begin{aligned}
&|P\big(T_0 \le c_{W_0}(\alpha)\big) - \alpha|\\
\le&|P\big(T_0 \le c_{W_0}(\alpha)\big) - P\big(T_0 \le c_{Z_0}(\alpha)\big)| + \rho\\
\le&P\Big(\{c_{Z_0}(\alpha) < T_0 \le c_{W_0}(\alpha)\} \cup \{c_{W_0}(\alpha) < T_0 \le c_{Z_0}(\alpha)\}\Big) + \rho\\
\le&\rho_\ominus + \rho.
\end{aligned}$$

Notice that $\Delta = O(n^{-1/2})$ when $\vartheta = O(n^{-1/4})$. The proof of Lemma S.3 is thus completed. □

With Lemma S.3, we next present the proof of Theorem 2.

Proof of Theorem 2: Define $T_{01}^* = F_{\mathrm{IE}}(X;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta_s,\Theta_s)$ and $\Delta_{T0} = F_{\mathrm{IE}}(0;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta,\Theta)$. It follows that $T_0^* = T_{01}^* + \Delta_{T0}$. Notice that

$$\rho^*(z) = \Big|P\{T_0^* \le z\} - P\{W_0^* \le z\}\Big|$$

$$
\begin{aligned}
&\le \Big|P\left\{T_0^* \le z\right\} - P\left\{T_0 \le z\right\}\Big| + \Big|P\left\{T_0 \le z\right\} - P\left\{W_0 \le z\right\}\Big| \\
&\qquad\qquad + \Big|P\left\{W_0^* \le z\right\} - P\left\{W_0 \le z\right\}\Big| \\
&\le \Big|P\left\{T_0^* \le z\right\} - P\left\{T_{01}^* \le z\right\}\Big| + \Big|P\left\{T_{01}^* \le z\right\} - P\left\{T_0 \le z\right\}\Big| \\
&\quad + \Big|P\left\{T_0 \le z\right\} - P\left\{W_0 \le z\right\}\Big| + \Big|P\left\{W_0^* \le z\right\} - P\left\{W_0 \le z\right\}\Big|.
\end{aligned}
$$

Similar to the proof of Lemma S.3, we have

$$
\begin{aligned}
&P\left\{T_{01}^* \le z\right\} - P\left\{T_0 \le z\right\} \\
\le &\big\{\mathbb{E}g\big(F_{\mathrm{IE}}(X;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta_s,\Theta_s)\big) - \mathbb{E}g\big(F_{\mathrm{IE}}(X;\theta,\Theta) - F_{\mathrm{IE}}(0;\theta,\Theta)\big)\big\} \\
&\quad + \big\{P(T_0 \le t + \psi^{-1}) - P(T_0 \le t)\big\} \\
= &\big\{\mathbb{E}g\big(F_{\mathrm{IE}}(X;\theta_s,\Theta_s) - F_{\mathrm{IE}}(X;\theta,\Theta)\big) - \mathbb{E}g\big(F_{\mathrm{IE}}(0;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta,\Theta)\big)\big\} \\
&\quad + \big\{P(T_0 \le t + \psi^{-1}) - P(T_0 \le t)\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&P\left\{W_0^* \le z\right\} - P\left\{W_0 \le z\right\} \\
\le &\big\{\mathbb{E}g\big(F_{\mathrm{IE}}(W;\widetilde{\theta},\widetilde{\Theta}) - F_{\mathrm{IE}}(0;\widetilde{\theta},\widetilde{\Theta})\big) - \mathbb{E}g\big(F_{\mathrm{IE}}(W;\theta,\Theta) - F_{\mathrm{IE}}(0;\theta,\Theta)\big)\big\} \\
&\quad + \big\{P(W_0 \le t + \psi^{-1}) - P(W_0 \le t)\big\} \\
= &\big\{\mathbb{E}g\big(F_{\mathrm{IE}}(W;\widetilde{\theta},\widetilde{\Theta}) - F_{\mathrm{IE}}(W;\theta,\Theta)\big) - \mathbb{E}g\big(F_{\mathrm{IE}}(0;\widetilde{\theta},\widetilde{\Theta}) - F_{\mathrm{IE}}(0;\theta,\Theta)\big)\big\} \\
&\quad + \big\{P(W_0 \le t + \psi^{-1}) - P(W_0 \le t)\big\}.
\end{aligned}
$$

Denote $\delta_{\theta s} = \theta_s - \theta$, $\delta_{\Theta s} = \Theta_s - \Theta$, $\delta_{\widetilde{\theta}} = \widetilde{\theta} - \theta$, and $\delta_{\widetilde{\theta}} = \widetilde{\theta} - \theta$. To bound these differences, the biases $\delta_{\theta s}, \delta_{\Theta s}, \delta_{\widetilde{\theta}}, \delta_{\widetilde{\theta}}$ can be treated in the same position as X or W. Take $F_{\mathrm{IE}}(W;\widetilde{\theta},\widetilde{\Theta})$ as an instance, we have

$$
\begin{aligned}
&F_{\mathrm{IE}}(W;\widetilde{\theta},\widetilde{\Theta}) \\
=&\frac{1}{m}\sum_{l=2}^{m}\left[\left(\beta(l) + \delta_{\widetilde{\beta}}(l) + \frac{e_l^{\beta}}{\sqrt{n}}\right)^{\top}\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}\left(\Phi(k) + \delta_{\widetilde{\Phi}}(k) + \frac{E_k^{\Phi}}{\sqrt{n}}\right)\left(\Gamma(j) + \delta_{\widetilde{\Gamma}}(j) + \frac{E_j^{\Gamma}}{\sqrt{n}}\right)\right\}\right].
\end{aligned}
$$

According to Theorem 1, $\delta_{\widetilde{\theta}}$ and $\delta_{\widetilde{\theta}}$ are asymptotic normal with variance of order n^{-1} (mean is negligible compared to the variance), i.e., that same order as $e_l^{\theta}/\sqrt{n}$. Hence by the same techniques as in proof of Lemma S.3, one can obtain

$$
\Big|P\left\{W_0^* \le z\right\} - P\left\{W_0 \le z\right\}\Big| \le Cn^{-1/8}.
$$

The biases $\delta_{\theta s}, \delta_{\Theta s}$ are of order $O(h^2 + m^{-1}) = o(n^{-1/2})$. They are not random given m and h. Then $\max_k \|\delta_{\Theta s}(k)\|_\infty \asymp \max_k \|\delta_{\theta s}(k)\|_\infty = o(n^{-1/2})$. Using similar arguments in proving Lemma S.3, we can show that

$$
\Big|P\left\{T_{01}^* \le z\right\} - P\left\{T_0 \le z\right\}\Big| \le Cn^{-1/8}.
$$

We omit the details to save space.

Finally, it remains to bound Δ_{T0}. Notice that

$$
\begin{aligned}
\Delta_{T0} &= F_{\mathrm{IE}}(0;\theta_s,\Theta_s) - F_{\mathrm{IE}}(0;\theta,\Theta) \\
&= \frac{1}{m}\sum_{l=2}^{m}\left[(\beta(l) + \delta_{\beta s}(l))^{\top}\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}(\Phi(k) + \delta_{\Phi s}(k))\,(\Gamma(j) + \delta_{\Gamma s}(j))\right\}\right] \\
&\qquad - \frac{1}{m}\sum_{l=2}^{m}\left[\beta(l)^{\top}\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}\Phi(k)\Gamma(j)\right\}\right] \\
&= \frac{1}{m}\sum_{l=2}^{m}\left[\beta(l)^{\top}\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}(\Phi(k) + \delta_{\Phi s}(k))\,(\Gamma(j) + \delta_{\Gamma s}(j)) - \prod_{k=j+1}^{l-1}\Phi(k)\Gamma(j)\right\}\right]
\end{aligned}
$$

$$
+\frac{1}{m}\sum_{l=2}^{m}\left[\delta_{\beta s}(l)^{\top}\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}\left(\Phi(k)+\delta_{\Phi s}(k)\right)\left(\Gamma(j)+\delta_{\Gamma s}(j)\right)\right\}\right].
$$

Let $\delta=\max\{\max_k\|\delta_{\theta s}(k)\|_\infty,\max_k\|\delta_{\Theta s}(k)\|_\infty\}=O(h^2+m^{-1})$. It follows that

$$
\begin{aligned}
&\left|\sum_{j=1}^{l-1}\left\{\prod_{k=j+1}^{l-1}\left(\Phi(k)+\delta_{\Phi s}(k)\right)\left(\Gamma(j)+\delta_{\Gamma s}(j)\right)-\prod_{k=j+1}^{l-1}\Phi(k)\Gamma(j)\right\}\right|\\
\le& M_\Gamma\sum_{j=1}^{l-1}\left|\prod_{k=j+1}^{l-1}\left(\Phi(k)+\delta\right)-\prod_{k=j+1}^{l-1}\Phi(k)\right|+\sum_{j=1}^{l-1}\prod_{k=j+1}^{l-1}\left|\left(\Phi(k)+\delta\right)\delta_{\Gamma s}(j)\right|\\
\lesssim&\sum_{j=1}^{l-1}\left|\sum_{k=1}^{l-1-j}\delta^k\binom{l-1-j}{k}q^{l-1-j-k}\right|+\delta\sum_{j=1}^{l-1}\prod_{k=j+1}^{l-1}\bar{q}\\
=&\sum_{j=1}^{l-1}\left|(\delta+q)^{l-1-j}-q^{l-1-j}\right|+\delta=\sum_{j=1}^{l-1}\{(\delta+q)^{l-1-j}-q^{l-1-j}\}+\delta\\
\lesssim&\left|\frac{1-q^l}{1-q}-\frac{1-(q+\delta)^l}{1-q-\delta}\right|+\delta\lesssim\delta\lesssim h^2+m^{-1}.
\end{aligned}
$$

Then, we have $\Delta_{T0}=O(h^2+m^{-1})$. Hence, $\left|P\left\{T_0^*\le z\right\}-P\left\{T_{01}^*\le z\right\}\right|=\left|P\left\{T_{01}^*+\Delta_{T0}\le z\right\}-P\left\{T_{01}^*\le z\right\}\right|\le P\{z-|\Delta_{T0}|\le T_{01}^*\le z+|\Delta_{T0}|\}=O(n^{1/2}h^2+n^{1/2}m^{-1})$ holds with probability 1 as $n\to\infty$. The proof is hence completed. □

S.7. Proof of Theorems 3, 4, Corollary 1 and More on the Switchback Design

Proof of Theorem 3: From Model 1, we can derive that for $i=1,\dots,n$, $t=1,\dots,m$,

$$
S_t=\Lambda_{t-1}^*+\mathbb{B}_{0t}S_1+\sum_{j=1}^{t-1}\mathbb{B}_{jt}\Gamma_jA_j+\varepsilon_{t-1S}^*, \tag{S.20}
$$

where $\mathbb{B}_{jt}=\prod_{k=j+1}^{t-1}\Phi(k)$, $\Lambda_{t-1}^*=\sum_{j=1}^{t}\mathbb{B}_{jt}\phi_0(j)$ and $\varepsilon_{t-1S}^*=\sum_{j=1}^{t-1}\mathbb{B}_{jt}\varepsilon_{jS}$. Define

$$
M_t^c=\begin{pmatrix}1 & A_{1t} & S_{1t}\\ 1 & A_{2t} & S_{2t}\\ \vdots & \vdots & \vdots\\ 1 & A_{nt} & S_{nt}\end{pmatrix},
$$

$$
\varphi_t=\frac{\mathbb{E}A_tS_t-\mathbb{E}A_t\mathbb{E}S_t}{1-\mathbb{E}A_t}=\frac{\mathrm{Cov}(A_t,S_t)}{\mathrm{Var}(A_t)/\mathbb{E}A_t}\quad\text{and}\quad\mathcal{S}_{0t}=\left[\mathrm{Var}\left(S_t-\frac{1}{\mathbb{E}A_t}\varphi_tA_t\right)\right]^{-1}.
$$

Under the two designs, A_t depends on the observed data only through the past actions. It follows from (S.20) that

$$
\varphi_t=2\sum_{j=1}^{t-1}\mathbb{B}_{jt}\Gamma_j\mathrm{Cov}(A_j,A_t),
$$

and hence

$$
\mathcal{S}_{0t}=\{\mathrm{Var}(\mathbb{B}_{0t}S_{i1}+\varepsilon_{i,t-1,S}^*)\}^{-1}.
$$

Direct algebra gives

$$
\left\{\frac{1}{n}(M_t^c)^\top M_t^c\right\}^{-1}=\begin{pmatrix}1 & \mathbb{E}A_t & \mathbb{E}S_t\\ \mathbb{E}A_t & \mathbb{E}A_t & \mathbb{E}A_tS_t\\ \mathbb{E}S_t & \mathbb{E}A_tS_t & \mathbb{E}S_t^2\end{pmatrix}^{-1}+o_p(1). \tag{S.21}
$$

Consider the matrix on the right-hand-side. Notice that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & -\varphi_t/\mathbb{E}A_t & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -\mathbb{E}A_t & 1 & \\ -\mathbb{E}S_t & & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathbb{E}A_t & \mathbb{E}S_t \\ \mathbb{E}A_t & \mathbb{E}A_t & \mathbb{E}A_tS_t \\ \mathbb{E}S_t & \mathbb{E}A_tS_t & \mathbb{E}S_t^2 \end{pmatrix}\begin{pmatrix} 1 & -\mathbb{E}A_t & -\mathbb{E}S_t \\ & 1 & \\ & & 1 \end{pmatrix}$$
$$\times\begin{pmatrix} 1 & & \\ & 1 & -\varphi_t/\mathbb{E}A_t \\ & & 1 \end{pmatrix}=\begin{pmatrix} 1 & & \\ & \mathrm{Var}(A_t) & \\ & & \mathrm{Var}[S_t-\mathrm{Var}^{-1}(A_t)\mathrm{Cov}(A_t,S_t)A_t] \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & \mathbb{E}A_t & \mathbb{E}S_t \\ \mathbb{E}A_t & \mathbb{E}A_t & \mathbb{E}A_tS_t \\ \mathbb{E}S_t & \mathbb{E}A_tS_t & \mathbb{E}S_t^2 \end{pmatrix}^{-1}=\begin{pmatrix} 1 & -\mathbb{E}A_t & -\mathbb{E}S_t \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & -\varphi_t/\mathbb{E}A_t \\ & & 1 \end{pmatrix}$$
$$\times\begin{pmatrix} 1 & & \\ & \frac{1}{(\mathbb{E}A_t)(1-\mathbb{E}A_t)} & \\ & & \mathcal{S}_{0,t} \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & -\varphi_t/\mathbb{E}A_t & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -\mathbb{E}A_t & 1 & \\ -\mathbb{E}S_t & & 1 \end{pmatrix}.$$

With some calculations, it follows from (S.21) that

$$\left\{\frac{1}{n}(M_t^c)^\top M_t^c\right\}^{-1}=\begin{pmatrix} * & * & * \\ -\frac{1}{1-\mathbb{E}A_t}-\frac{\varphi_t\mathcal{S}_{0t}(\varphi_t-\mathbb{E}S_t)}{\mathbb{E}A_t} & \frac{1}{\mathbb{E}A_t(1-\mathbb{E}A_t)}+\frac{\varphi_t^2\mathcal{S}_{0t}}{\mathbb{E}^2A_t} & -\frac{\varphi_t\mathcal{S}_{0t}}{\mathbb{E}A_t} \\ \mathcal{S}_{0t}\left(\varphi_t-\mathbb{E}S_t\right) & -\frac{\mathcal{S}_{0t}\varphi_t}{\mathbb{E}A_t} & \mathcal{S}_{0t} \end{pmatrix}+o_p(1).$$

Consequently, the resulting OLS estimator satisfies

$$\begin{aligned}
\widehat{\gamma}(t)=&\gamma(t)+\frac{1}{n}\left\{-\frac{1}{1-\mathbb{E}A_t}-\frac{\varphi_t\mathcal{S}_{0t}(-\mathbb{E}S_t+\varphi_t)}{\mathbb{E}A_t}\right\}\sum_i e_{it}\\
&+\frac{1}{n}\left\{\frac{1}{\mathbb{E}A_t(1-\mathbb{E}A_t)}+\frac{\varphi_t^2\mathcal{S}_{0t}}{\mathbb{E}^2A_t}\right\}\sum_i A_{it}e_{it}-\frac{1}{n}\frac{\varphi_t\mathcal{S}_{0t}}{\mathbb{E}A_t}\sum_i S_{it}e_{it}+o_p(n^{-1/2})\\
=&\gamma(t)+\frac{1}{n}\left\{\frac{1}{\mathbb{E}A_t(1-\mathbb{E}A_t)}+\frac{\varphi_t^2\mathcal{S}_{0t}}{\mathbb{E}^2A_t}\right\}\sum_i(A_{it}-\mathbb{E}A_t)e_{it}-\frac{1}{n}\frac{\varphi_t\mathcal{S}_{0t}}{\mathbb{E}A_t}\sum_i(S_{it}-\mathbb{E}S_t)e_{it}+o_p(n^{-1/2})\\
=&\gamma(t)+\frac{4}{n}\sum_{i=1}^n(A_{it}-\mathbb{E}A_t)e_{it}-\frac{2}{n}\varphi_t\mathcal{S}_{0t}\sum_{i=1}^n\{\mathbb{B}_{0t}(S_{i1}-\mathbb{E}S_1)+\varepsilon_{i,t-1,S}^*\}e_{it}+o_p(n^{-1/2})\\
&+\frac{4}{n}\varphi_t^2\mathcal{S}_{0t}\sum_{i=1}^n(A_{it}-\mathbb{E}A_t)e_{it}-\frac{2}{n}\varphi_t\mathcal{S}_{0t}\sum_{j=1}^{t-1}\mathbb{B}_{jt}\Gamma_j\sum_{i=1}^n(A_{ij}-\mathbb{E}A_j)e_{it}.
\end{aligned}$$

It can be verified that under either the alternating-day or the switchback design, the last line equals 0. Specifically, under the alternating-day design, we have $A_{ij}=A_{it}$. Hence, $\varphi_t=2^{-1}\sum_{j=1}^{t-1}\mathbb{B}_{jt}\Gamma_j$ and the last term becomes $4n^{-1}\varphi_t^2\mathcal{S}_{0t}\sum_{i=1}^n(A_{it}-\mathbb{E}A_t)e_{it}$, equal to the first term on the last line. Under the switchback design, noting the relationship $A_{ij}-\mathbb{E}A_j=(-1)^{t-j}(A_{it}-\mathbb{E}A_t)$, we can obtain that the last line equals 0. Hence, we have

$$\widehat{\gamma}(t)=\gamma(t)+\frac{4}{n}\sum_{i=1}^n(A_{it}-\mathbb{E}A_t)e_{it}-\frac{2}{n}\varphi_t^\top\mathcal{S}_{0t}\sum_{i=1}^n\{\mathbb{B}_{0t}(S_{i1}-\mathbb{E}S_{i1})+\varepsilon_{i,t-1,S}^*\}e_{it}+o_p(n^{-1/2}).$$

This gives

$$\widehat{\mathrm{DE}}=\mathrm{DE}+\frac{4}{n}\sum_{t=1}^m\sum_{i=1}^n(A_{it}-\mathbb{E}A_t)e_{it}-\frac{2}{n}\sum_{t=2}^m\left[\varphi_t\mathcal{S}_{0t}\sum_{i=1}^n\{\mathbb{B}_{0t}(S_{i1}-\mathbb{E}S_{i1})+\varepsilon_{i,t-1,S}^*\}e_{it}\right]+o_p(n^{-1/2}).$$

Then

$$MSE(\widehat{\mathrm{DE}})=\frac{1}{n}\mathbb{E}\left\{4\sum_{t=1}^m(A_{it}-\mathbb{E}A_t)e_{it}\right\}^2+\frac{1}{n}\mathbb{E}\left[2\sum_{t=2}^m\varphi_t\mathcal{S}_{0t}\{\mathbb{B}_{0t}(S_{i1}-\mathbb{E}S_{i1})+\varepsilon_{i,t-1,S}^*\}e_{it}\right]^2.$$

In the switchback design, $A_{i1}=1-A_{i2}=\cdots=A_{i,\tau-1}=1-A_{i\tau}$. Denote $E_{it}=\mathcal{S}_{0t}\{\mathbb{B}_{0t}(S_{i1}-\mathbb{E}S_{i1})+\varepsilon^*_{i,t-1,S}\}e_{it}$. Then we have

$$
\begin{aligned}
&nMSE(\widehat{\mathrm{DE}}_{ad})-nMSE(\widehat{\mathrm{DE}}_{sb})\\
=&4\mathrm{Var}\left(\sum_{k=1}^{m}e_k\right)-4\mathrm{Var}\left\{\sum_{k=1}^{m/2}(e_{2k-1}-e_{2k})\right\}+\mathbb{E}\left[2\sum_{t=2}^{m}\varphi_{t,ad}E_{it}\right]^2-\mathbb{E}\left[2\sum_{t=2}^{m}\varphi_{t,sb}E_{it}\right]^2+o(1)\\
=&4\sum_{j,k}\Sigma_e(j,k)-4\sum_{j,k}(-1)^{|j-k|}\Sigma_e(j,k)+4\sum_{t_1,t_2=2}^{m}\mathrm{Cov}(E_{it_1},E_{it_2})(\varphi_{t_1,ad}-\varphi_{t_1,sb})(\varphi_{t_2,ad}+\varphi_{t_2,sb})+o(1).
\end{aligned}
$$

Under the given conditions, E_{it_1} and E_{it_2} are positively correlated. When $\{\Phi(\tau)\}_\tau$ and $\{\Gamma(\tau)\}_\tau$ are of the same signs, respectively, we have

$$
\begin{aligned}
&\sum_{t_1,t_2=2}^{m}\mathrm{Cov}(E_{it_1},E_{it_2})(\varphi_{t_1,ad}-\varphi_{t_1,sb})(\varphi_{t_2,ad}+\varphi_{t_2,sb})\\
=&\sum_{t_1,t_2=2}^{m}\mathrm{Cov}(E_{it_1},E_{it_2})\left(\sum_{\substack{1\le j_1\le t_1-1\\ t_1-j_1=1,3,\ldots}}\prod_{k=j_1+1}^{t_1-1}\Phi(k)\Gamma(j_1)\right)\left(\sum_{\substack{1\le j_2\le t_2-1\\ t_2-j_2=2,4,\ldots}}\prod_{k=j_2+1}^{t_2-1}\Phi(k)\Gamma(j_2)\right)\ge 0.
\end{aligned}
$$

Hence

$$
nMSE(\widehat{\mathrm{DE}}_{ad})-nMSE(\widehat{\mathrm{DE}}_{sb})\ge 8\sum_{|j-k|=1,3,\ldots}\Sigma_e(j,k)+o(1).
$$

This completes the proof. □

Proof of Corollary 1: Without loss of generality, assume the constant c equals one. With some calculations, we have that

$$
\begin{aligned}
MSE(\widehat{\mathrm{DE}}_{sb})&\asymp\frac{4}{n}\sum_{j,k}\sum_{j,k}(-1)^{|j-k|}\Sigma_e(j,k)\\
&=4n^{-1}\left\{m+2\sum_{k=1}^{m/2}(m-2k)\rho^{2k}-2\sum_{k=1}^{m/2}(m-2k+1)\rho^{2k-1}\right\}\\
&=4n^{-1}\left\{m-2\sum_{k=1}^{m/2}(m-2k)(\rho^{2k-1}-\rho^{2k})-2\sum_{k=1}^{m/2}\rho^{2k-1}\right\}\\
&=4n^{-1}\left\{m-2(1-\rho)\sum_{k=1}^{m/2}(m-2k)\rho^{2k-1}\right\}\\
&=4n^{-1}\left\{m-2m(1-\rho)\sum_{k=1}^{m/2}\rho^{2k-1}\right\}=\frac{1-\rho}{1+\rho}4n^{-1}m,\\
MSE(\widehat{\mathrm{DE}}_{ad})&\asymp\frac{4}{n}\Sigma_e(j,k)\\
&=4n^{-1}\left\{m+2\sum_{k=1}^{m/2}(m-2k)\rho^{2k}+2\sum_{k=1}^{m/2}(m-2k+1)\rho^{2k-1}\right\}\\
&=4n^{-1}\left\{m+2m(1+\rho)\sum_{k=1}^{m/2}\rho^{2k-1}\right\}+o(n^{-1})=\frac{1+\rho}{1-\rho}4n^{-1}m,
\end{aligned}
$$

which yields that $\mathrm{MSE}(\widehat{\mathrm{DE}}_{sb})/\mathrm{MSE}(\widehat{\mathrm{DE}}_{ad})\asymp(1-\rho)^2/(1+\rho)^2$. □

Proof of Theorem 4: When $m=2$, IE essentially equals $\beta(2)\Gamma(1)$. It follows that

$$
\widehat{\mathrm{IE}}=\mathrm{IE}+[\widehat{\beta}(2)-\beta(2)]\Gamma(1)+\beta(2)[\widehat{\Gamma}(1)-\Gamma(1)]+o_p(n^{-1/2}).
$$

Similar to the proof of Theorem 3, it can be shown that

$$\begin{aligned}
\widehat{\beta}(2)-\beta(2) &= \frac{\mathcal{S}_{02}}{n}\sum_{i=1}^{n}(S_{i2}-\mathbb{E}S_2)e_{i2}-\frac{2\mathcal{S}_{02}\varphi_2}{n}\sum_{i=1}^{n}(A_{i2}-\mathbb{E}A_2)e_{i2}+o_p(n^{-1/2}) \\
&= \frac{\mathcal{S}_{02}}{n}\sum_{i=1}^{n}[\Phi(1)(S_{i,1}-\mathbb{E}S_1)+\Gamma(1)(A_{i,1}-\mathbb{E}A_1)+\varepsilon_{i,1S}]e_{i2} \\
&- \frac{2\mathcal{S}_{02}\varphi_2}{n}\sum_{i=1}^{n}(A_{i2}-\mathbb{E}A_2)e_{i2}+o_p(n^{-1/2}). \\
\widehat{\Gamma}(1)-\Gamma(1) &= \frac{4}{n}\sum_{i=1}^{n}(A_{i1}-\mathbb{E}A_1)\varepsilon_{i,1S}+o_p(n^{-1/2}). \qquad \text{(S.22)}
\end{aligned}$$

Under both designs, we can show that

$$\widehat{\beta}(2)-\beta(2)=\frac{\mathcal{S}_{02}}{n}\sum_{i=1}^{n}[\Phi(1)(S_{i,1}-\mathbb{E}S_1)+\varepsilon_{i,1S}]e_{i2}+o_p(n^{-1/2}). \qquad \text{(S.23)}$$

Notice that the i.i.d. sums in both (S.22) and (S.23) are design independent. Consequently, the IE estimators under the two designs achieve the same asymptotic MSE. The proof is hence completed. □

Comparison against the regular switchback design. We next compare our switchback design against the regular switchback design (Bojinov et al., 2020) which administers independent Bernoulli treatments across time. Consider the case where there exists some $0<\rho<1$ such that for any $1\le j,k\le m$, $\mathrm{Cov}(\eta_j,\eta_k)=\Sigma_{\eta,jk}=\rho^{|j-k|}$, $\mathrm{Var}(\varepsilon_j)=\{\sigma_j^2\}_j$. It follows that $\mathrm{Cov}(e_j,e_k)=\rho^{|j-k|}+\sigma_j^2\mathbb{I}\{j=k\}$. We focus on the variance of DE under the settings of Corollary 1. We first calculate the covariance of highest resolution covariance Σ_e with $\rho=0.8$, $\sigma_j^2=0.36$ and $m=144$, and then generate covariances of $m=72,48,36,24,12,6$ by computing the corresponding sub-matrices from Σ_e. As shown in Figure S.1 below, the proposed switchback design is more efficient than the regular one for any m. It also implies that the variances decrease with m in both designs.

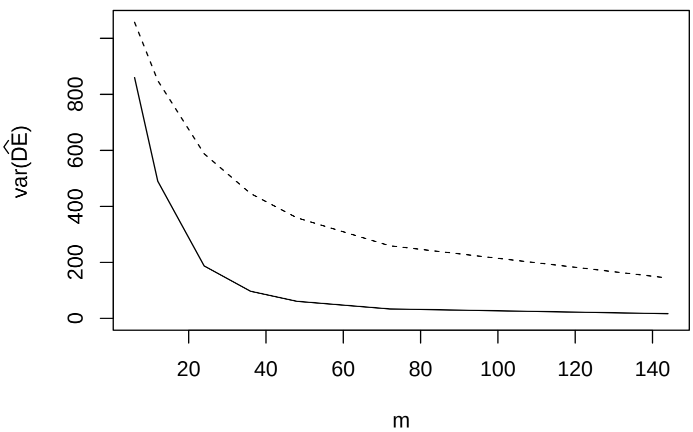

Fig. S.1. The solid line represents the variance of the DE under the proposed switchback design and whereas the dash line represents the one under the regular switchback design.

S.8. Proof of Theorem 5

We first establish the error bound for $|\widehat{\mathrm{DE}}-\mathrm{DE}|$. Recall that

$$\widehat{\mathrm{DE}}-\mathrm{DE}=\frac{1}{nM}\sum_{i=1}^{n}\sum_{k=1}^{M}\sum_{\tau=1}^{m}\mathbb{E}^*\left[\left\{\widehat{g}_1(\tau,\widehat{S}^0_{i\tau k})-\widehat{g}_0(\tau,\widehat{S}^0_{i\tau k})\right\}-\mathbb{E}\left\{g_1(\tau,S^0_\tau)-g_0(\tau,S^0_\tau)\right\}\right].$$

It follows that

$$
\begin{aligned}
&|\widehat{\mathrm{DE}}-\mathrm{DE}| \\
\leq&\sum_{a=0}^{1}\left|\sum_{\tau=1}^{m}\frac{1}{nM}\sum_{i=1}^{n}\sum_{k=1}^{M}\left\{\widehat{g}_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right\}\right| \\
=&\sum_{a=0}^{1}\left|\sum_{\tau=1}^{m}\frac{1}{nM}\sum_{i=1}^{n}\sum_{k=1}^{M}\left\{\widehat{g}_a(\tau,\widehat{S}_{i\tau k}^0)-g_a(\tau,\widehat{S}_{i\tau k}^0)\right\}+\frac{1}{nM}\sum_{\tau=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{M}\left\{g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right\}\right| \qquad \text{(S.24)}\\
\leq&\sum_{a=0}^{1}\sum_{\tau=1}^{m}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left\{\widehat{g}_a(\tau,\widehat{S}_{i\tau k}^0)-g_a(\tau,\widehat{S}_{i\tau k}^0)\right\}\right|+\sum_{a=0}^{1}\sum_{\tau=1}^{m}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right| \\
&+O_p\big(\sqrt{m}(nM)^{-1/2}\big)
\end{aligned}
$$

where the expectation $\mathbb{E}^*$ is taken with respect to the simulated random errors.

We next calculate the bound $\left|n^{-1}\sum_{i=1}^n\left\{\mathbb{E}^*g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right\}\right|$ for $1\leq\tau\leq m$, $a=0,1$. Notice that for $\tau\geq 2$, the density of S_τ^0 conditional on $S_{\tau-1}^0$ can be expressed as $f_{\varepsilon_\tau S}\left(s-G_0(\tau-1,S_{\tau-1}^0)\right)$, and the density of $\widehat{S}_{i\tau k}^0$ is $\widehat{f}_{\varepsilon_\tau S}\left(s-\widehat{G}_0(\tau-1,\widehat{S}_{i,\tau-1,k}^0)\right)$. We next derive the bound of

$$\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^*\left\{g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right\}\right|,$$

for $1\leq\tau\leq m$.

- When $\tau=1$, we have $\widehat{S}_{i1k}^0=S_{i1}$. Then $n^{-1}\sum_{i=1}^n\mathbb{E}^*g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)=n^{-1}\sum_{i=1}^n g_a(\tau,S_{i1})-\mathbb{E}g_a(\tau,S_1)$, where S_{i1} and S_1 are identically distributed. According to Hoeffding's inequality, the difference is
$$O_p(n^{-1/2}\sqrt{\log m+\log n}).$$
- When $\tau=2$, by definition, we have
$$\mathbb{E}g_a(2,S_2^0)=\mathbb{E}\int_s g_a(2,s)f_{\varepsilon_2 S}(s-G_0(1,S_1))ds,$$
and that
$$\mathbb{E}^*g_a(2,\widehat{S}_{i,2,k}^0)=\int_s g_a(2,s)\widehat{f}_{\varepsilon_2 S}(s-\widehat{G}_0(1,S_{i,1}))ds.$$
With some calculations, we have
$$
\begin{aligned}
&\left|n^{-1}\sum_{i=1}^{n}\mathbb{E}^*\left\{g_a(\tau,\widehat{S}_{i\tau k}^0)-\mathbb{E}g_a(\tau,S_\tau^0)\right\}\right| \\
=&\mathbb{E}\int_s g_a(2,s)\left|f_{\varepsilon_2 S}(s-G_0(1,S_1))-\widehat{f}_{\varepsilon_2 S}(s-\widehat{G}_0(1,S_1))\right|ds+O(n^{-1/2}\sqrt{\log(mn)}) \\
\leq&\mathbb{E}\int_s g_a(2,s)|f_{\varepsilon_2 S}(s-G_0(1,S_{i,1}))-f_{\varepsilon_2 S}(s-\widehat{G}_0(1,S_{i,1}))|ds \\
+&\mathbb{E}\int_s g_a(2,s)|\widehat{f}_{\varepsilon_2 S}(s-\widehat{G}_0(1,S_{i,1}))-f_{\varepsilon_2 S}(s-\widehat{G}_0(1,S_{i,1}))|ds+O(n^{-1/2}\sqrt{\log(mn)}).
\end{aligned}
$$
Under the given condition, the second last line is upper bounded by $L_f\mathbb{E}|G_0(1,S_{i,1})-\widehat{G}_0(1,S_{i,1})|\leq L_f\Delta_1(n,m)$, using Cauchy-Schwarz inequality. Additionally, the first term on the last line is upper bounded by $O_p(\Delta_3(n,m))$.
- More generally, when $\tau\geq 3$, we have
$$\mathbb{E}g_a(\tau,S_\tau^0)=\mathbb{E}\int_{s_\tau,s_{\tau-1},\cdots,s_2}g_a(\tau,s_\tau)f_{\varepsilon_\tau S}(s_\tau-G_0(\tau-1,s_{\tau-1}))\cdots f_{\varepsilon_2 S}(s_2-G_0(1,S_1))ds_\tau\cdots ds_2,$$
and that
$$\mathbb{E}^*g_a(\tau,\widehat{S}_{i,\tau,k}^0)=\int_{s_\tau,s_{\tau-1},\cdots,s_2}g_a(\tau,s_\tau)\widehat{f}_{\varepsilon_\tau S}(s_\tau-\widehat{G}_0(\tau-1,s_{\tau-1}))\cdots\widehat{f}_{\varepsilon_2 S}(s_2-\widehat{G}_0(1,S_{i,1}))ds_\tau\cdots ds_2.$$

Similarly, we can show that the difference $\left|n^{-1}\sum_{i=1}^n \mathbb{E}^*\left\{g_a(\tau,\widehat{S}^0_{i\tau k})-\mathbb{E}g_a(\tau,S^0_\tau)\right\}\right| = \sum_{j=2}^\tau \zeta_{1,j}+\zeta_{2,j}+O_p(n^{-1/2}\sqrt{\log(mn)})$, where $\zeta_{1,j}$ is given by

$$\mathbb{E}\int_{s_\tau,\cdots,s_2} g_a(\tau,s_\tau)\widehat{f}_{\varepsilon_\tau S}(s_\tau-\widehat{G}_0(\tau-1,s_{\tau-1}))\cdots|\widehat{f}_{\varepsilon_j S}(s_j-\widehat{G}_0(j-1,s_{j-1}))-f_{\varepsilon_j S}(s_j-\widehat{G}_0(j-1,s_{j-1}))|$$
$$\times f_{\varepsilon_{j-1}S}(s_{j-1}-G_0(j-2,s_{j-2}))\cdots f_{\varepsilon_2 S}(s_2-G_0(1,S_{i,1}))ds_\tau\cdots ds_2,$$

whose absolute value can be upper bounded by

$$O(1)\mathbb{E}\int_{s_j,\cdots,s_2}|\widehat{f}_{\varepsilon_j S}(s_j-\widehat{G}_0(j-1,s_{j-1}))-f_{\varepsilon_j S}(s_j-\widehat{G}_0(j-1,s_{j-1}))|$$
$$\times f_{\varepsilon_{j-1}S}(s_{j-1}-G_0(j-2,s_{j-2}))\cdots f_{\varepsilon_2 S}(s_2-G_0(1,S_{i,1}))ds_\tau\cdots ds_2=O(\Delta_3(n,m)),$$

and $\zeta_{2,j}$ is given by

$$\mathbb{E}\int_{s_\tau,\cdots,s_2} g_a(\tau,s_\tau)\widehat{f}_{\varepsilon_\tau S}(s_\tau-\widehat{G}_0(\tau-1,s_{\tau-1}))\cdots|f_{\varepsilon_j S}(s_j-\widehat{G}_0(j-1,s_{j-1}))-f_{\varepsilon_j S}(s_j-G_0(j-1,s_{j-1}))|$$
$$\times f_{\varepsilon_{j-1}S}(s_{j-1}-G_0(j-2,s_{j-2}))\cdots f_{\varepsilon_2 S}(s_2-G_0(1,S_{i,1}))ds_\tau\cdots ds_2,$$

whose absolute value can be upper bounded by

$$O(1)L_f\mathbb{E}|\widehat{G}_0(j-1,S^0_{j-1})-G_0(j-1,S^0_{j-1})|\le O(1)L_f\sqrt{\mathbb{E}|\widehat{G}_0(j-1,S^0_{j-1})-G_0(j-1,S^0_{j-1})|^2},$$

where S^0_{j-1} denotes the potential state assuming the system receives the control treatment at each time. Using the change of measure theory, we obtain that $\zeta_{2,j}=O_p(L_f\sqrt{\omega}\Delta_1(n,m))$ where the big-O_p term is uniform in j.

To summarize, we obtain that

$$\left|\frac{1}{n}\sum_{i=1}^n\mathbb{E}^*\left\{g_a(\tau,\widehat{S}^0_{i\tau k})-\mathbb{E}g_a(\tau,S^0_\tau)\right\}\right|=O_p(n^{-1/2}\sqrt{\log m+\log n}+\tau\Delta_3(n,m)+L_f\sqrt{\omega}\tau\Delta_1(n,m)).$$

We next bound $\left|n^{-1}\sum_{i=1}^n\mathbb{E}^*\left\{\widehat{g}_a(\tau,\widehat{S}^0_{i\tau k})-g_a(\tau,\widehat{S}^0_{i\tau k})\right\}\right|$. Notice that

$$\left|\frac{1}{n}\sum_{i=1}^n\mathbb{E}^*\left\{\widehat{g}_a(\tau,\widehat{S}^0_{i\tau k})-g_a(\tau,\widehat{S}^0_{i\tau k})\right\}\right|$$
$$\le\mathbb{E}\left|\widehat{g}_a(\tau,S^0_\tau)-g_a(\tau,S^0_\tau)\right|+\left|\frac{1}{n}\sum_{i=1}^n\mathbb{E}^*(\widehat{g}_a-g_a)(\tau,\widehat{S}^0_{i\tau k})-\mathbb{E}(\widehat{g}_a-g_a)(\tau,S^0_\tau)\right|.$$

Similarly, we can show that the second term is $O_p(n^{-1/2}\sqrt{\log(mn)}+L_f\tau\sqrt{\omega}\Delta_1(n,m)+\tau\Delta_3(n,m))$ whereas the first term can be upper bounded by $O(\sqrt{\omega}\Delta_2(n,m))$ using the change of measure theorem. Consequently, we have shown that

$$|\widehat{\mathrm{DE}}-\mathrm{DE}|=O_p(mn^{-1/2}\sqrt{\log(nm)}+m^2\Delta_3(n,m)+L_f m^2\sqrt{\omega}\Delta_1(n,m)+m\sqrt{\omega}\Delta_2(n,m)).$$

As for the error bound for $|\widehat{\mathrm{IE}}-\mathrm{IE}|$, it can be expressed by

$$\left|\widehat{\mathrm{IE}}-\mathrm{IE}\right|=\left|\frac{1}{nM}\sum_{i=1}^n\sum_{k=1}^M\sum_{\tau=1}^m\left[\left\{\widehat{g}_1(\tau,\widehat{S}^1_{i\tau k})-\widehat{g}_1(\tau,\widehat{S}^0_{i\tau k})\right\}-\mathbb{E}\left\{g_1(\tau,S^1_\tau)-g_1(\tau,S^0_\tau)\right\}\right]\right|$$
$$\le\sum_{\tau=1}^m\left[\left|\frac{1}{nM}\sum_{i=1}^n\sum_{k=1}^M\widehat{g}_1(\tau,\widehat{S}^1_{i\tau k})-\mathbb{E}\left\{g_1(\tau,S^1_\tau)\right\}\right|+\left|\frac{1}{nM}\sum_{i=1}^n\sum_{k=1}^M\widehat{g}_1(\tau,\widehat{S}^0_{i\tau k})-\mathbb{E}\left\{g_1(\tau,S^0_\tau)\right\}\right|\right].$$

The error bound can be obtained using similar arguments in deriving the error bound of $|\widehat{\mathrm{DE}}-\mathrm{DE}|$. We omit the details to save space.

Table S.1. Simulation results of DE test based on temporal model and data from city A. We report the rejection probabilities of 400 replicates for different temporal-alternating design of experiment ($TI = 1, 3, 6$), number of days ($n = 8, 14, 20$), and relative improvement in percentage ($\delta = 0.00, 0.25, 0.50, 0.75, 1.00$).

y	$hour$	n	0.00	0.25	0.50	0.75	1.00
DTI	1	8	6.8	24.2	47.2	64.2	76.0
		14	6.5	34.2	65.0	82.0	91.0
		20	5.5	38.2	74.8	90.2	96.2
	3	8	6.5	15.8	33.0	47.2	62.2
		14	3.8	19.0	42.5	64.2	78.5
		20	5.2	26.0	53.0	77.0	91.5
	6	8	6.8	12.5	18.2	29.8	40.8
		14	6.8	12.0	23.5	37.8	49.5
		20	6.8	13.0	28.8	46.0	61.8

S.9. Proofs of Theorems 6 and 7

The proofs of Theorems 6 and 7 are very similar to those of Theorems 1 and 2, and we sketch an outline only. To prove the consistency of the proposed test for DE in Theorem 6, it suffices to show the joint asymptotic normality of the set of estimated varying coefficients $\{\widetilde{\theta}_{st}(\tau,\iota)\}_{\tau,\iota}$. We first notice that, the initial estimator obtained in Step 1 of Algorithm S.1 is obtained by applying Steps 1 and 2 of Algorithm 1 to each individual region. The asymptotic normality of the initial estimator can be proven using similar arguments in the proof of Theorem 1.

Next, note that the refined estimator $(\widetilde{\theta}(1,\iota)^\top, \ldots, \widetilde{\theta}(1,\iota)^\top)^\top$ is essentially a linear transformation of the initial estimator. Using similar arguments in Section S.5, we can further calculate the asymptotic bias and variance, as well as the asymptotic normality of $\widetilde{\theta}_{st}(\tau,\iota)$, based on the expression $\widetilde{\theta}_{st}(\tau,\iota) = \kappa_{\ell,h_{st}}(\iota)\widetilde{\theta}^0_{st}(\tau,\ell)$.

The proof of Theorem 7 is similar to that of Theorem 2. The only difference lies in the dimension of parameter vector. To be specific, let $e_i^\beta(\tau,\iota), E_i^\Phi(\tau,\iota), E_i^\Gamma(\tau,\iota)$ be the analogs of $e_i^\beta(\tau)$, $E_i^\Phi(\tau), E_i^\Gamma(\tau)$ for $1 \le \tau \le m$, $1 \le \iota \le r$ under the spatiotemporal case. Denote

$$
\begin{aligned}
x_i^{st}(\tau,\iota) &= \left(e_i^\beta(\tau,\iota)^\top, \{\mathrm{vec}(E_i^\Phi(\tau,\iota))\}^\top, E_i^\Gamma(\tau,\iota)^\top\right)^\top \in \mathbb{R}^{2d(d+2)},\\
x_i^{st}(\iota) &= \left(x_i(2,\iota)^\top, x_i(3,\iota)^\top, \ldots, x_i(m,\iota)^\top\right)^\top \in \mathbb{R}^{p_x}, \quad p_x = 2(m-1)dp,\\
x_i^{st} &= \left(x_i^{st}(1)^\top, x_i^{st}(2)^\top, \ldots, x_i^{st}(r)^\top\right)^\top \in \mathbb{R}^{p_x^{st}}, \quad p_x^{st} = 2(m-1)dpr.
\end{aligned} \tag{S.25}
$$

Define the function

$$
\begin{aligned}
F_{\mathrm{IE}}^{st} = \frac{1}{mr}\sum_{\iota=1}^{r}\sum_{\tau=2}^{m}\Bigg[&\left(\beta_s(\tau,\iota) + \frac{e^\beta_{\tau,\iota}}{\sqrt{n}}\right)^\top\\
&\cdot\sum_{j=1}^{\tau-1}\left\{\prod_{k=j+1}^{\tau-1}\left(\Phi_s(k,\iota) + \frac{E^\Phi_{k,\iota}}{\sqrt{n}}\right)\left(\Gamma_s(j,\iota) + \frac{E^\Gamma_{j,\iota}}{\sqrt{n}}\right)\right\}\Bigg].
\end{aligned}
$$

Similar to Theorem 2, the proof of Theorem 7 contains two steps. In the first step, we could employ the high-dimensional Gaussian approximation theory to bound the difference between $\widehat{\mathrm{IE}}_{st} - \mathrm{IE}_{st}$ and $\widehat{\mathrm{IE}}^b_{st} - \widehat{\mathrm{IE}}_{st}$, assuming that these statistics are constructed based on the oracle parameters. This allows us to establish the validity of the bootstrap algorithm in the second step. As we have commented, the only difference lies in the dimension of parameters, and the results can be derived similarly using the arguments in the proof for Theorem 2.

Table S.2. Simulation results of DE test based on temporal model and data from city B. We report the rejection probabilities of 400 replicates for different temporal-alternating design of experiment ($hour = 1, 3, 6$), number of days ($n = 8, 14, 20$), and relative improvement in percentage ($\delta = 0.00, 0.25, 0.50, 0.75, 1.00$).

y	$hour$	n	0.00	0.25	0.50	0.75	1.00
DTI	1	8	4.0	14.5	29.5	49.8	64.8
		14	4.0	21.5	50.0	79.0	93.2
		20	3.5	22.8	62.2	86.0	97.0
	3	8	4.2	8.5	17.0	26.8	35.8
		14	4.2	11.5	22.8	35.2	51.0
		20	7.2	15.3	31.0	46.8	60.5
	6	8	7.8	11.2	17.5	23.2	28.8
		14	6.8	10.5	18.8	28.0	37.2
		20	7.2	15.2	23.0	31.5	45.5

Table S.3. Simulation results of IE test based on temporal model and data from city A.

TI	n	0	0.25	0.5	0.75	1
1	8	4.8	12.0	46.5	74.8	87.0
	14	6.0	25.5	75.2	89.8	94.5
	20	6.2	47.0	86.8	93.8	97.0
3	8	4.8	10.0	21.5	46.8	64.5
	14	6.2	21.8	49.5	72.8	84.2
	20	6.0	23.8	66.0	83.0	89.2
6	8	5.0	9.2	17.0	32.5	52.0
	14	5.8	15.5	37.8	65.2	77.0
	20	5.8	22.0	58.2	76.5	83.5

Table S.4. Simulation results of IE test based on temporal model and data from city B.

TI	n	0	0.25	0.5	0.75	1
1	8	5.2	9.2	32.8	64.8	80.0
	14	5.5	18.2	66.0	83.5	91.2
	20	7.2	33.5	79.5	91.2	95.5
3	8	5.0	8.5	15.5	30.2	52.8
	14	5.5	17.5	33.5	62.5	75.0
	20	5.8	19.5	52.0	75.0	85.5
6	8	5.0	7.8	13.5	21.8	34.8
	14	6.5	13.8	23.5	50.0	68.0
	20	5.5	15.2	36.5	65.5	77.5

Table S.5. Simulation results of DE test based on spatiotemporal model and data from city A.

		Temporal-alternating					
	DE	0	0.5		1		
	delta1	0	0	0.5	0	0.5	1
	delta2	0	0.5	0	1	0.5	0
TI=1	n=8	5.0	41.3	50.8	60.5	65.3	82.8
	n=14	5.3	55.5	70.3	74.0	87.3	94.0
	n=20	3.8	70.8	82.3	85.8	94.0	96.3
TI=3	n=8	4.8	33.0	36.8	56.8	59.0	65.5
	n=14	5.0	40.8	48.8	75.5	77.0	85.5
	n=20	4.0	57.0	65.8	80.5	81.3	90.8
TI=6	n=8	4.0	17.5	21.0	19.3	21.3	33.3
	n=14	3.5	28.3	34.5	27.5	43.8	49.5
	n=20	6.0	31.8	39.0	48.5	50.3	54.8
		Spatiotempotal-alternating					
	DE	0	0.5		1		
	delta1	0	0	0.5	0	0.5	1
	delta2	0	0.5	0	1	0.5	0
TI=1	n=8	5.0	46.0	56.3	67.3	68.8	85.0
	n=14	6.3	62.3	75.5	81.0	91.0	97.3
	n=20	5.3	76.0	87.3	92.0	97.5	100.0
TI=3	n=8	4.3	38.3	44.0	62.5	62.5	68.0
	n=14	8.5	47.3	54.3	81.5	81.5	88.5
	n=20	6.5	61.8	71.0	85.3	85.3	92.8
TI=6	n=8	2.8	23.0	28.3	25.3	26.5	37.8
	n=14	4.5	34.3	41.3	34.3	50.3	55.8
	n=20	5.8	37.3	44.8	53.5	57.5	62.3

Table S.6. Simulation results of IE test based on spatiotemporal model and data from city A.

		Temporal-alternating					
	IE	0	0.5		1		
	delta1	0	0	0.5	0	0.5	1
	delta2	0	0.5	0	1	0.5	0
TI=1	n=8	6.0	57.3	63.8	83.8	92.8	94.0
	n=14	5.3	76.0	78.0	92.0	94.3	97.0
	n=20	4.0	88.8	90.8	94.3	96.3	98.3
TI=3	n=8	4.5	45.0	49.5	53.3	60.5	68.0
	n=14	5.3	60.5	61.8	64.0	69.5	84.8
	n=20	3.5	75.8	77.0	72.3	84.5	92.3
TI=6	n=8	6.0	29.8	32.0	50.8	61.3	63.8
	n=14	4.8	50.5	51.0	59.0	68.0	82.5
	n=20	4.8	59.5	61.5	77.5	83.5	88.3
		Spatiotempotal-alternating					
	IE	0	0.5		1		
	delta1	0	0	0.5	0	0.5	1
	delta2	0	0.5	0	1	0.5	0
TI=1	n=8	4.3	59.3	66.0	85.8	94.3	96.0
	n=14	6.3	78.5	80.3	93.0	96.0	98.0
	n=20	6.5	90.0	92.0	95.8	97.5	99.8
TI=3	n=8	5.0	47.0	51.5	55.0	62.0	70.0
	n=14	5.5	62.0	63.8	65.8	71.5	85.8
	n=20	5.3	77.0	78.8	73.3	86.3	93.5
TI=6	n=8	6.0	31.3	34.0	51.8	62.3	64.8
	n=14	4.8	52.0	53.3	61.3	70.5	84.3
	n=20	4.8	62.0	63.0	79.8	86.0	90.3

S.10. Tables and Figures

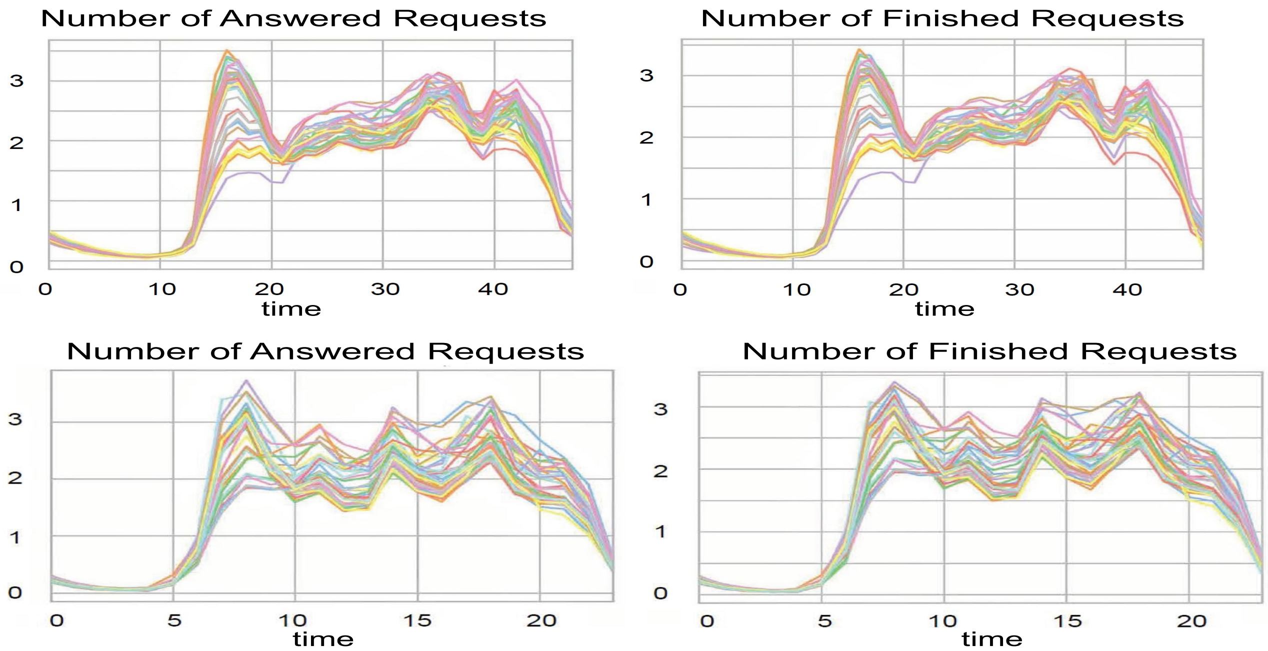

Fig. S.2. Scaled numbers of answered and finished requests from City A (the first row) and City B (the second row) across 40 days .

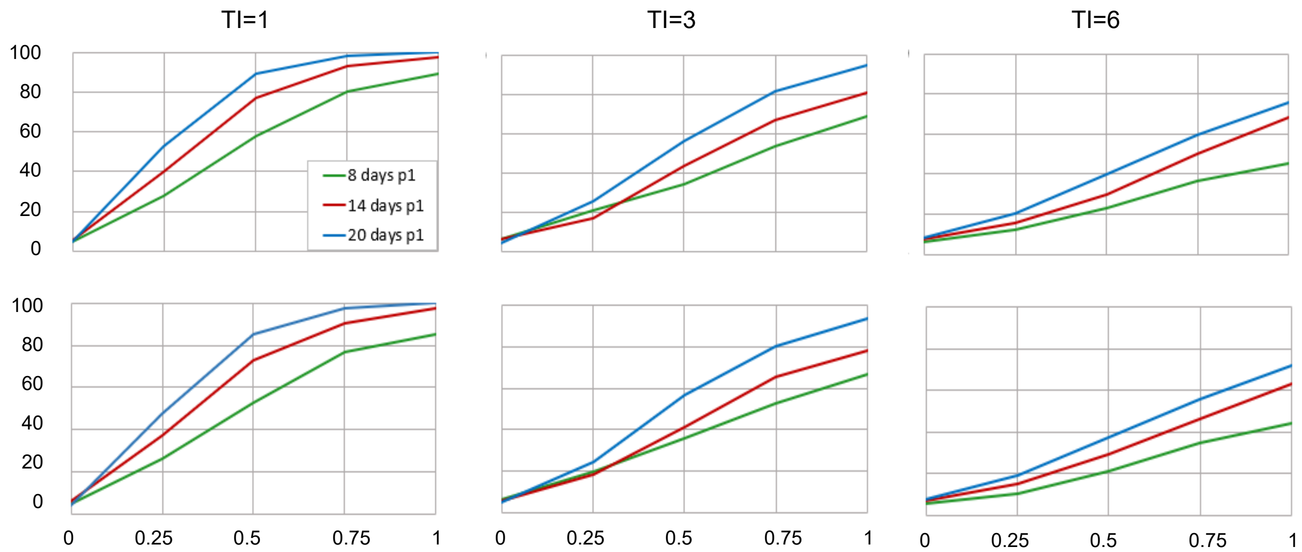

Fig. S.3. Empirical rejection rates of the proposed test for DE, with different combinations of n, δ, TI and outcomes based on the real dataset from city A (the number of answered requests in the first row and the number of finished requests in the second row).

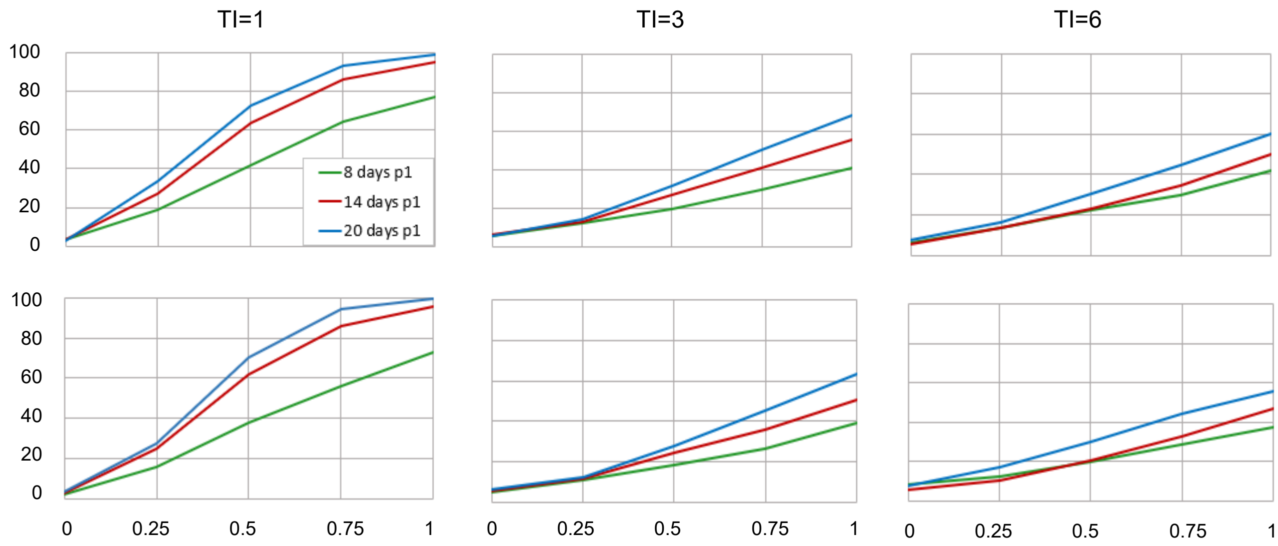

Fig. S.4. Empirical rejection rates of the proposed test for DE, with different combinations of n, δ, TI and outcomes based on the real dataset from city B (the number of answered requests in the first row and the number of finished requests in the second row).

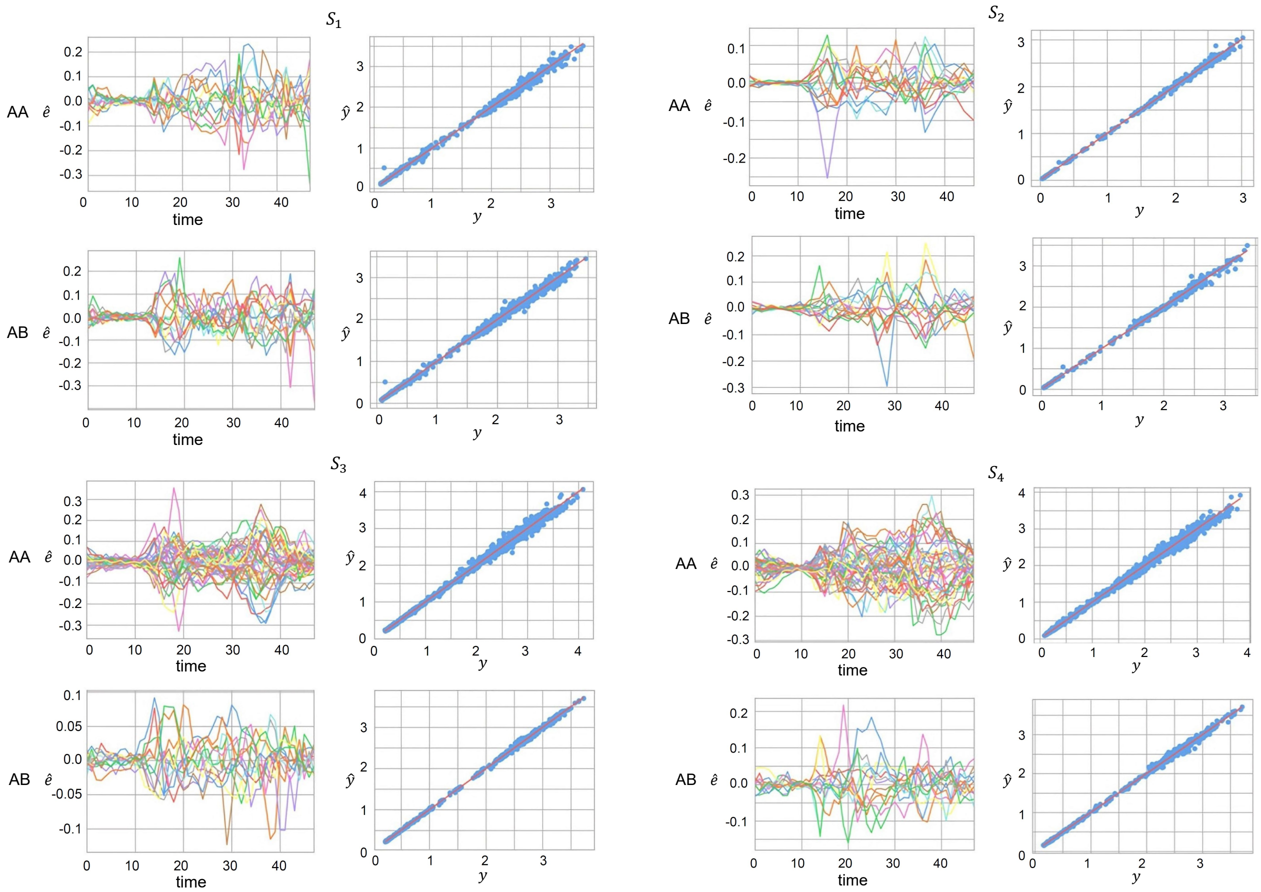

Fig. S.5. Plots of the fitted drivers' total income against the observed values as well as the corresponding residuals. Data are collected from an A/A or A/B experiment under the temporal alternation design.

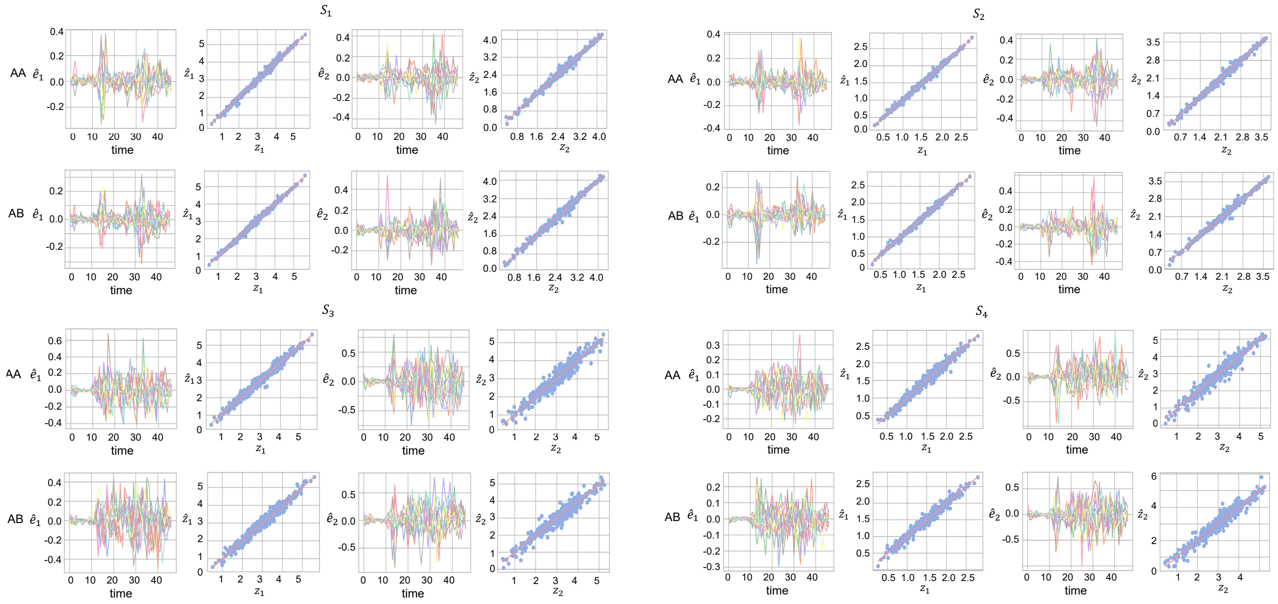

Fig. S.6. Plots of the fitted number of orders ($\hat{e}_1$) and drivers' online time ($\hat{e}_2$) against their observed values, as well as the corresponding residuals. Data are collected from an A/A or A/B experiment under the temporal alternation design.

S.11. Codes

S.11.1. Code for cross validation

```
import numpy as np
import pandas as pd
import statsmodels.api as sm
import statsmodels.formula.api as smf
from itertools import product
import multiprocessing as mp
import os
import warnings
warnings.filterwarnings("ignore")
import sys
path='../temporal/src'
if path not in sys.path:
    sys.path.append(path)
from sklearn.model_selection import KFold
from model_new import VCM

### simulation settings ###
file = 'V2_hangzhou_serial_order_dispatch_AA.csv'
ycol = 'gmv'
xcols = ['cnt_call','sum_online_time']
scols = ['cnt_call_1','sum_online_time_1']
acol = 'is_exp'
regcols = ['const'] + xcols

df = pd.read_csv('C:/Users/annie/OneDrive - pku.edu.cn/projects/3. Finished/
                                        stvcm/Code+Data20210825/temporal/data/'
                                        +file)
df['const'] = 1
xycols = [ycol] + regcols +['date', 'time']
df = df[xycols]

NN = 40
idx = [i+1 for i in range(NN)]
```

```
kf = KFold(n_splits=5, shuffle=True)

param_grid = [0.05*i for i in range(20)] * NN ** (-1/3)

K = 3; M =48

res = []

for train_index, test_index in kf.split(idx):
    df_train = df.loc[df['date'].isin(train_index)].set_index(['date','time'])
    df_test = df.loc[df['date'].isin(test_index)].set_index(['date','time'])
    for hc in param_grid:
        Amat = df_train.groupby('date').apply(lambda dt: np.dot(dt[regcols].T.
                                               values, dt[regcols].values)).
                                               sum()
        bvec = df_train.groupby('date').apply(lambda dt: np.dot(dt[regcols].T,
                                                dt[ycol])).sum()
        eps_diag = np.eye(Amat.shape[0])*1e-3
        theta = np.linalg.solve(Amat+eps_diag, bvec)
        theta = pd.DataFrame(theta.reshape((M, K)), columns=regcols)
        tmat = np.mat(np.reshape(np.repeat(np.arange(M)/(M-1), M), (M,M)))
        theta = smooth(theta.T, ker_mat((tmat.T-tmat),hc)).T
        df_test['fitted'] = df_test[regcols].dot(theta_DE.values.flatten())
        df_test['resid'] = df_test[ycol] - df_test['fitted']
        res.append(sum((df_test['resid'])**2))

res = np.array(res).reshape(5,20)
res = res.sum(axis=0)

np.array(param_grid)[np.where(np.min(res))]
```

S.11.2. Main code

```
import numpy as np
import pandas as pd
import statsmodels.api as sm
import statsmodels.formula.api as smf
from itertools import product
import multiprocessing as mp
from numpy import kron
import os
import warnings
warnings.filterwarnings("ignore")
import sys
path='.../Spatio-temporal/src'
if path not in sys.path:
    sys.path.append(path)
from model_st_new import VCM

### simulation settings ###
ycol = 'gmv'#,'cnt_grab', 'cnt_finish']
xcol = 'cnt_call'#,'sum_online_time']
scol = 'cnt_call_1'#the lag term
acol = 'is_exp'
acol_n = 'is_exp_n'
regcols = ['const'] + [xcol]
adj_mat = np.array([[0,1,1,1,0,0,0,0,0,0],
          [1,0,0,1,1,0,0,0,0,0],
          [1,0,0,1,0,1,0,0,0,0],
          [1,1,1,0,1,1,1,0,0,0],
          [0,1,0,1,0,0,1,1,0,0],
```

```
            [0,0,1,1,0,0,1,0,1,0],
            [0,0,0,1,1,1,0,1,1,1],
            [0,0,0,0,1,0,1,0,0,1],
            [0,0,0,0,0,1,1,0,0,1],
            [0,0,0,0,0,0,1,1,1,0]])
G = 10
adj_mat = adj_mat/np.repeat(adj_mat.sum(axis=0),G).reshape(G,G)
nsim = 400

two_sided = False
wild_bootstrap = False
interaction = False

DDS = [0.00, 0.005, 0.01]
IIS = [0.00, 0.005, 0.01]
IIS_n = [0.00, 0.005, 0.01]
NNs = [8,14,20]
TIs = [1,3,6]
designs = ['st','t']

wbi = 1 if wild_bootstrap else 0
tsi = 1 if two_sided else 0
ini = 1 if interaction else 0
hc = 0.01
hc_b = 0.01
IE = True

DD = 0
for (II, II_n, TI, design, NN) in product(IIS, IIS_n, TIs, designs, NNs):

    file = 'V1_hangzhou_pool.csv'
    df = pd.read_csv('../data/'+file, index_col=['grid_id','date','time'])
    path = '../res/IE_{}_{}_{}_{}_{}.npy'.format(design, file, NN, TI, DDS)
    if os.path.exists(path):
        continue

    df['const'] = 1
    M = len(df.index.get_level_values(2).unique())
    N = len(df.index.get_level_values(1).unique())
    NM = M*N
    if IE:
        df[scol] = np.append(np.delete(df[xcol].values
        *(df.index.get_level_values(2)>0),0),0)
        df[scol][df[scol]==0] = np.nan
        xyscols = [ycol] + regcols + [scol]
        df = df[xyscols]
    else:
        xycols = [ycol] + regcols
        df = df[xycols]
    df[acol] = -1

    model0 = VCM(df, ycol, xcol, acol, scol,IE,
                     interaction=interaction,
                     two_sided=two_sided,
                     wild_bootstrap=wild_bootstrap,
                    center_x=True, scale_x=True,hc=hc)
    model0.estimate(null = True)
    df['fitted_DE'] = model0.holder['fitted_DE'].values
    df['eta_DE'] = model0.holder['eta_DE'].values
    df['eps_DE'] = model0.holder['eps_DE'].values
    df['fitted_IE'] = model0.holder['fitted_IE'].values
    df['eta_IE'] = model0.holder['eta_IE'].value
```

```
df['eps_IE'] = model0.holder['eps_IE'].values

def generate(df, N, ycol, regcols, acol, ti=1, delta=0, delta_s=0,
                                        delta_s_n=0):
    grids = (df.index.get_level_values(0).unique())
    G = len(grids)
    dates = (df.index.get_level_values(1).unique())
    number_of_days = len(dates)
    M = len(df)// G // number_of_days

    dates_ = np.random.choice(dates, size=(N,), replace=True)
    df_ = df.loc[[(x,y,z) for x in grids for y in dates_ for z in range(M)
                                            ],:]
    df_ = df_.reset_index()
    df_['date'] = np.tile(np.repeat(np.arange(N),M), G)
    df_.set_index(['grid_id','date','time'], inplace=True)

    mt = int(24/ti)
    if ti < 24: # intra-day time interval
        abv = np.tile(np.repeat([-1,1], M//mt), mt//2)
        bav = np.tile(np.repeat([1,-1], M//mt), mt//2)
        vec = np.hstack([abv, bav])
    elif ti == 24: # inter-day time interval
        av = -np.ones(M)
        bv = np.ones(M)
        vec = np.hstack([av, bv])
    gvs = np.array([])
    gv = np.tile(vec, N//2)
    if design == 'st':
        for i in range(G):
            gvs = np.append(gvs, np.random.choice([-1,1])*gv)
    else:
        for i in range(G):
            gvs = np.append(gvs, gv)
    df_[acol] = gvs
    df[acol_n] = np.dot(adj_mat, ((df[acol].values+1)/2).reshape(G,M*N)).
                                            ravel()

    if IE:
        idx1 = np.arange(df_.shape[0])[df_.index.get_level_values(2)>0]
        a=(df_['fitted_IE'] + \
                    df_['eps_IE'] * np.repeat(np.random.randn(N*G), M) + \
                            df_['eta_IE'] * np.repeat(np.random.randn(N*G)
                                                                , M)).
                                                                values
        df_[xcol].iloc[idx1]=a[~np.isnan(a)]
        df_[xcol] *= (1+delta_s_n)
        df_.loc[df_[acol]==1, xcol] *= (1+delta_s)
        df_[scol] = np.append(np.delete(df_[xcol].values
        *(df_.index.get_level_values(2)>0),0),0)
        df_[scol][df_[scol]==0] = np.nan
    df_[ycol] = (df_['fitted_DE'] + \
                    df_['eps_DE'] * np.repeat(np.random.randn(N*G), M) + \
                    df_['eta_DE'] * np.repeat(np.random.randn(N*G), M)).
                                                        values
    df_[ycol] *= (1+delta_s_n)
    df_.loc[df_[acol]==1, ycol] *= (1+delta+delta_s)

    return df_

def one_step(seed):
```

```
        np.random.seed(seed)
        ret = []

        df_ = generate(df, NN, ycol, regcols, acol, TI, DD, II, II_n)
        model = VCM(df_, ycol, xcol, acol, acol_n, scol,IE,
                    interaction=interaction,
                    two_sided=two_sided,
                    wild_bootstrap=wild_bootstrap,
                   center_x=True, scale_x=True,hc=hc)
        if IE==0:
            model.inference()
            ret.append([model.holder['test_stats_wb'], model.holder['test_stat
                                                              '],
                    model.holder['pvalue1'], model.holder['pvalue2']])
        else:
            model.estimate()
            ret.append(model.holder['test_stat_IE'])

        return ret

    pool = mp.Pool(20)
    rets = pool.map(one_step, range(nsim))
    rets = np.array(rets)
    pool.close()

    path = '../res/IE_{}_{}_{}_{}_{}.npy'.format(design, file, NN, TI, DDS)

    np.save(path, rets)
```

S.12. Further Discussions and Extensions

S.12.1. Endogeneity bias

In this subsection, we discuss how to remove the endogeneity bias when the random effects appear in the state regression model as well. Specifically, Model 1 becomes

$$
\begin{aligned}
Y_{i,\tau} &= \beta_0(\tau) + S_{i,\tau}^\top \beta(\tau) + A_{i,\tau}\gamma(\tau) + e_{i,\tau} = Z_{i,\tau}^\top \theta(\tau) + e_{i,\tau},\\
S_{i,\tau+1} &= \phi_0(\tau) + \Phi(\tau) S_{i,\tau} + A_{i,\tau}\Gamma(\tau) + e_{i,\tau S} = \Theta(\tau) Z_{i,\tau} + e_{i,\tau S},
\end{aligned}
$$

where $e_{i,\tau S} = \eta_{i,\tau S} + \varepsilon_{i,\tau S}$, $\eta_{i,\tau S}$ characterizes the day-specific temporal variation across different days and $\varepsilon_{i,\tau S}$ is the measurement error. We assume that $\eta_{i,\tau S}, \varepsilon_{i,\tau S}$ are mutually independent; $\{\varepsilon_{i,\tau S}\}_{i,\tau}$ are independent measurement errors with zero means and $\mathrm{Cov}(\varepsilon_{i,\tau S}) = \Sigma_{\varepsilon,\tau S}$; and $\{\eta_{i,\tau S}\}_{i,\tau}$ are identical copies of a mean-zero stochastic process with covariance function and $\{\Sigma_{\eta_S}(\tau_1,\tau_2)\}_{\tau_1,\tau_2}$.

Due to the potential dependencies between these random effects, past and future features are no longer conditionally independent. Directly applying existing OPE methods or our proposal developed in Section 2 would yield biased policy value estimators. Note that the predictor $S_{i,\tau} = \Theta(\tau-1)Z_{i,\tau-1} + e_{i,\tau-1,S}$ at time τ is dependent upon the $e_{i,\tau}$ due to the existence of the random effects in these residuals, resulting in endogeneity in the state regression model. As a result, the resulting OLS estimator is biased, leading to inconsistent estimation of IE.

We next outline two approaches to remove the endogeneity bias. The first approach relies on the use of historical data in which the actions were the set to baseline policy. According to the state regression model, $\{S_t\}_t$ in the historical data satisfies

$$
S_{t+1} = \phi_0^*(t) + \Phi^*(k) S_1 + e_{tS}^*,
$$

where $\phi_0^*(t) = \sum_{k=1}^t \phi_0(k) \prod_{\ell=k+1}^t \Phi(\ell)$, $\Phi^*(k) = \prod_{k=1}^t \Phi(k)$ and the error e_{tS}^* is independent of S_1. As such, the OLS estimator $\widehat{\Phi}^*(k)$ is consistent. When $\{\Phi(k)\}_k$ are nonzero, it allows us to consistently estimate these regression coefficients. On the other hand, when the actions are independent of the states, the regression coefficients $\{\Gamma(\tau)\}_\tau$ can be consistently estimated using data collected from online experiments. This allows us to consistently estimate IE based on (7).

The second approach requires the random effects to satisfy certain covariance structures. In particular, we require the correlation between $\eta_{i,\tau_1 S}$ and $\eta_{i,\tau_2 S}$ to decay to zero as $|\tau_{1S} - \tau_{2S}|$ approaches infinity. For a given sufficiently large m_1, the residual error e_{tS} and the past state S_{t-m_1} become asymptotically uncorrelated. According to the state regression model, we obtain that

$$S_t = \phi(0) + \Phi(t) S_{t-m_1} + \sum_{k=t-m_1}^{t-1} \Gamma_t(k) A_k + e_{tS},$$

where $\Gamma_t(k) = (\Phi(t-1)\Phi(t-2)\ldots\Phi(k+1))\Gamma(k)$ and can be consistently estimated via OLS. As such, IE can be consistently estimated as well by noting that

$$\mathrm{IE} = \sum_{t=2}^{m} \beta(t)^\top \left\{ \sum_{k=1}^{t-1} \Phi(k) \left(\sum_{\ell=k-m_1}^{k-1} \Gamma_k(\ell) \right) \right\}.$$

S.12.2. High-dimensional models

We extend the proposed method to settings with high-dimensional state information in this section. For simplicity, we focus on the linear temporal varying coefficient model example. In the high-dimensional setting, we assume most elements in the regression coefficients $\beta(\tau)$ and $\Phi(\tau)$ are equal to zero. Hypothesis testing is challenging since many penalized estimators such as the Lasso (Tibshirani, 1996) or the Dantzig selector (Candes and Tao, 2007) does not have a traceable limiting distribution.

One solution is to employ regularization methods with folded-concave penalty functions such as the smoothly clipped absolute deviation (SCAD, Fan and Li, 2001), adaptive Lasso (Zou, 2006) or minimal concave penalty (MCP, Zhang, 2010) in Step 1 of Algorithms 1 and 2 to obtain sparse estimators. Under certain minimal-signal-strength assumptions, the resulting estimators possess the "oracle" property in that they are selection consistent and asymptotically equivalent to the oracle OLS estimators computed as if the supports were known in advance (Fan and Lv, 2011). As such, the proposed Wald-type test statistics for DE remain valid. The bootstrap procedure is equally applicable even when the number of parameters is much larger than the sample size (Dezeure et al., 2017; Zhang and Cheng, 2017). We may also apply sample splitting (Dezeure et al., 2015) or the recursive online-score estimation (ROSE) algorithm (Shi et al., 2021) to account for model selection uncertainty.

Another solution is to employ the debiasing method (Javanmard and Montanari, 2014; Van de Geer et al., 2014; Zhang and Zhang, 2014; Ning and Liu, 2017) to allow for valid inference without the minimal-signal-strength assumption. Specifically, we first apply penalized regression with LASSO, SCAD or MCP to obtain the initial regression estimators. We next debias these initial estimators using decorrelated estimation (see e.g. Shi and Li, 2021, Equation 14). This strategy guarantees each entry of the final estimator is asymptotically normal, regardless of whether the minimal-signal-strength assumption holds or not. These final estimators can be subsequently used for testing DE and IE.

S.12.3. Test Procedures based on the Unsmoothed Estimator

As commented in the main text, we can also use the unsmoothed estimators to test DE and IE. The resulting tests require weaker conditions on m compared to those built upon the smoothed estimators. Specifically, m is allowed to be either fixed, or to diverge to infinity. To the contrary, tests based on smoothed estimators require m to diverge with n at certain rate.

Test statistics based on the unsmoothed estimators are given by

$$\widetilde{\mathrm{DE}} = \sum_{\tau=1}^{m} \widehat{\gamma}(\tau), \quad \widetilde{\mathrm{IE}} = \sum_{\tau=2}^{m} \widehat{\beta}(\tau)^\top \left\{ \sum_{k=1}^{\tau-1} \left(\prod_{l=k+1}^{\tau-1} \widehat{\Phi}(l) \right) \widehat{\Gamma}(k) \right\}.$$

The standard error of $\widetilde{\mathrm{DE}}$ is computed based on $\widehat{\boldsymbol{V}}_\theta$ which we denote by $\widehat{se}(\widetilde{\mathrm{DE}})$. The residuals and pseudo-outcomes for computing bootstrap samples are also constructed based on the OLS estimators $\widehat{\theta}(\tau)$ and $\widehat{\Theta}(\tau)$. The following results follow immediately from Theorem 1(i).

PROPOSITION 3. *Suppose the assumptions in Theorem 1 hold. Then under H_0^{DE}, we have $\mathbb{P}(\widetilde{DE}/\widehat{se}(\widetilde{DE}) > z_\alpha) = \alpha + o(1)$; under H_1^{DE}, we have $\mathbb{P}(\widetilde{DE}/\widehat{se}(\widetilde{DE}) > z_\alpha) \to 1$.*

Similar to Theorem 2, we can show that the bootstrap procedure based on the unsmoothed estimators is valid to infer IE as well.

PROPOSITION 4. *Suppose that there exist some constants* $0 < c_1 \le 1, 0 \le c_2 < 3/2$ *such that* $c_1 \le \mathbb{E}\|\varepsilon_{\tau,S}\|^2$, $\mathbb{E}e_\tau^2 \le c_1^{-1}$ *for all* $1 \le \tau \le m$ *and that* $m = O(n^{c_2})$. *Suppose the assumptions in Theorem 1 as well as Assumptions 5 holds. Then, with probability approaching 1,*

$$\sup_z |\mathbb{P}(\widetilde{IE} - IE \le z) - \mathbb{P}(\widetilde{IE}^b - \widetilde{IE} \le z|Data)| \le Cn^{-1/8},$$

for some positive constant $C > 0$.

S.12.4. Advantage of the decomposition of DE and IE

Recall that the DE represents the sum of the short-term treatment effects on the immediate outcome over time assuming that the baseline policy is being employed in the past. In contrast, IE characterizes the carryover effects of past policies through their impact on the state variables (e.g., the demand and supply in the ridesharing platform).

Gaining insights into both DE and IE is instrumental in understanding the mechanisms through which the new policy surpasses the existing one, thereby paving the way for the creation of even more effective strategies. For instance, if the new policy's DE exceeds that of the current policy, but its IE is smaller, then adopting either policy in isolation would yield similar results on average. However, studying this decomposition enables us to derive a hybrid strategy that employs the existing policy during the first half of the day and switches to the new policy for the latter half. Given that DE characterizes short-term effects and IE measures delayed effects, it is reasonable to expect this hybrid approach to outperform both original policies. To see this, we use the temporal case as an instance and denote

$$\begin{aligned}
\mathrm{DE}_\tau &= \mathbb{E}\{R_\tau(1, S_\tau^*(\mathbf{0}_{\tau-1}), 0, S_{\tau-1}^*(\mathbf{0}_{\tau-2}), \ldots, S_1) - R_\tau(0, S_\tau^*(\mathbf{0}_{\tau-1}), 0, S_{\tau-1}^*(\mathbf{0}_{\tau-2}), \ldots, S_1)\}, \\
\mathrm{IE}_\tau &= \mathbb{E}\{R_\tau(1, S_\tau^*(\mathbf{1}_{\tau-1}), 1, S_{\tau-1}^*(\mathbf{1}_{\tau-2}), \ldots, S_1) - R_\tau(1, S_\tau^*(\mathbf{0}_{\tau-1}), 0, S_{\tau-1}^*(\mathbf{0}_{\tau-2}), \ldots, S_1)\}.
\end{aligned}$$

We remark that DE_τ represents the direct effect on R_τ of applying the new policy only during time interval τ and IE_τ represents the indirect effect on R_τ of applying the new policy from time interval 1 to $(\tau - 1)$. Denote $\mathrm{IE}_1 = 0$, $\overline{\mathrm{DE}} = (\mathrm{DE}_1, \mathrm{DE}_2, \ldots, \mathrm{DE}_m)^\top$ and $\overline{\mathrm{IE}} = (\mathrm{IE}_1, \mathrm{IE}_2, \ldots, \mathrm{IE}_m)^\top$. Then we have

$$\mathrm{DE} = \mathbf{1}_m^\top \overline{\mathrm{DE}} \quad \text{and} \quad \mathrm{IE} = \mathbf{1}_m^\top \overline{\mathrm{IE}}$$

where $\mathbf{1} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^m$. Suppose that we are interested in the policy effects of a specific time period $1 \le m_1 \le \tau \le m_2 \le m$ and denote the corresponding DE and IE by DE_{m_1,m_2} and IE_{m_1,m_2}. Let $\mathbf{1}_{m_1,m_2}$ be the m-dimensional vector whose $(m_1, m_1+1, \ldots, m_2)$th elements are 1 and the other elements are 0. Then

$$\mathrm{DE}_{m_1,m_2} = \mathbf{1}_{m_1,m_2}^\top \overline{\mathrm{DE}} \quad \text{and} \quad \mathrm{IE}_{m_1,m_2} = \mathbf{1}_{m_1,m_2}^\top \overline{\mathrm{IE}}.$$

Using the same technique of inferring DE and IE, we can test

$$\begin{aligned}
H_0(\mathrm{DE}_{m_1,m_2}) : \mathrm{DE}_{m_1,m_2} \le 0 \quad &\text{versus} \quad H_1(\mathrm{DE}_{m_1,m_2}) : \mathrm{DE}_{m_1,m_2} > 0; \\
H_0(\mathrm{IE}_{m_1,m_2}) : \mathrm{IE}_{m_1,m_2} \le 0 \quad &\text{versus} \quad H_1(\mathrm{IE}_{m_1,m_2}) : \mathrm{IE}_{m_1,m_2} > 0;
\end{aligned}$$

which can guide the strategy design. For instance, if $H_0(\mathrm{DE}_{m_1,m_2})$ is rejected and $H_0(\mathrm{IE}_{m_1,m_2})$ is not, we can apply the new policy from time m_1 and m_2 and keep the old policy from time 1 to $m_1 - 1$ to save the cost.

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