## §0 Introduction

Consider the curve $x^2+y^2=1$ in the space $\mathbb{R}^2$

rational solutions of $x^2+y^2=1 \xleftrightarrow{1:1}$ rational points on the curve

What ~~is~~ ~~the~~ are algebraic funtions on the curve $x^2+y^2=1$?

$\left\{\begin{matrix}\text{algebraic functions}\\ \text{on } \mathbb{R}^2\end{matrix}\right\}$ forms ~~a~~ the ring $\mathbb{R}[X,Y]$.

For $f,g\in\mathbb{R}[X,Y]$, if $f-g\in(X^2+y^2-1)$, then $f$ and $g$ define the same funtion on the cure $x^2+y^2=1$.

$\Rightarrow$ alg. functions on $x^2+y^2=1$ are related to $\dfrac{\mathbb{R}[X,Y]}{(X^2+y^2-1)}$.

Now consider the curve $x^2+y^2-1=0$ in $\mathbb{C}\times\mathbb{C} \rightsquigarrow \dfrac{\mathbb{C}[X,Y]}{(X^2+y^2-1)}$

point $(a,b)$ on $x^2+y^2-1=0 \longleftrightarrow$ maximal ideal $(X-a,Y-b)$

$? \longleftrightarrow$ prime ideals ?

$\left\{\begin{matrix}\text{geo.}\\ \text{points}\end{matrix}\right\} \subseteq \operatorname{Spec}\dfrac{\mathbb{C}[X,Y]}{(X^2+y^2-1)} = \{\text{prime ideals}\}$

with topology: Zariski top, closed subsets is defined by common zeros of some polynomials, i.e., $Z(T)=\{p\in$

More generally, we will arise a spectral space Spec $A$ with Zariski top.

Scheme = glue along such spectral spaces

"coarse manifold".

— (local) functions on this space Spec $A$ $\Big\langle$ $A_\mathfrak{p}$ localization
$A$

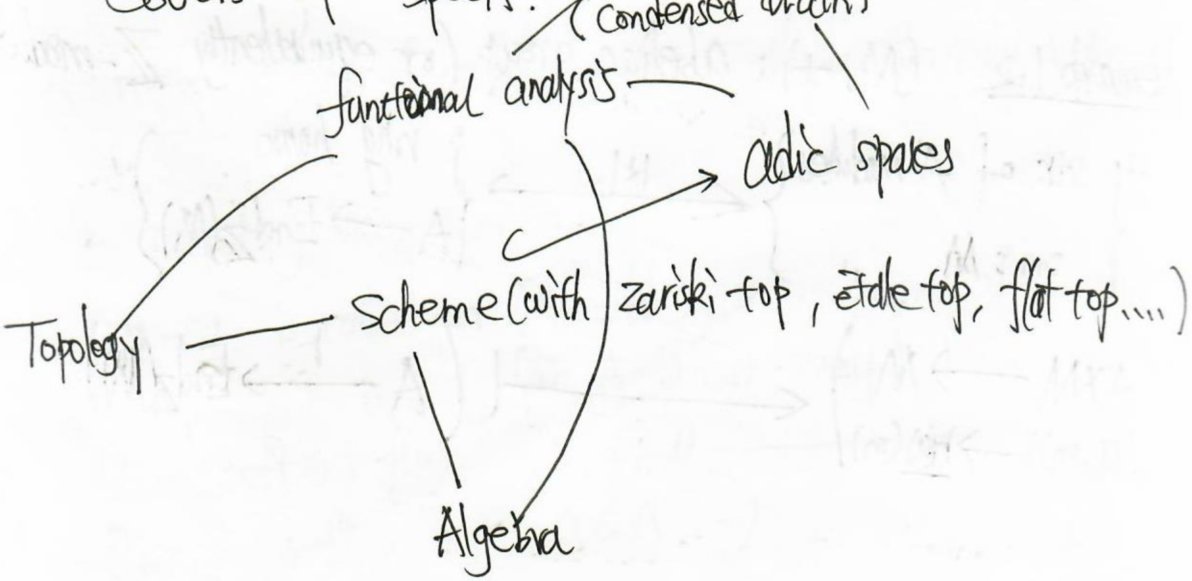
— More generally, can consider sheaves of modules on it,

for Spec $A$, it corresponds to $A$-modules.

— (relative) cohomology ⟶ derived functors

— dimension of Spec $A$

covers of Spec $A$.



# §1 Rings and Modules

(directed) diagram = points (objects)
( Category ) + directed edges (morphisms)
+ composition

**Definition 1.1** A: ring. An A-module is an abelian group (M,+ together with a linear A-action:

$$\begin{array}{rl} \mu: A\times M \longrightarrow M & \\ (a,x) \longmapsto a.x = \mu(a,x) & \end{array} \left( \Longleftrightarrow \begin{array}{l} \text{ring homo} \\ A \rightarrow \mathrm{End} \\ a \mapsto (\mu(a,- \end{array} \right.$$

satisfying

$$\begin{array}{ll} a(x+y) = ax+ay & \\ (a+b)x = ax+by & a,b\in A \\ (ab)x = a(bx) & x,y\in M. \\ 1\cdot x = x & \end{array}$$

**Remark 1.2** $(M,+)$: abelian group (or equivalently, $\mathbb{Z}$-module)

$$\left\{ \begin{array}{l} \text{str. of } A\text{-modules} \\ \text{on } M \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{ring homo} \\ A \longrightarrow \mathrm{End}_{\mathbb{Z}}(M) \end{array} \right\}$$

$$\left( \begin{array}{l} A\times M \longrightarrow M \\ (a,m) \longmapsto \underline{r(a)}(m) \end{array} \right) \longleftarrow\!\!\!\mid \left( A \xrightarrow{r} \mathrm{End}_{\mathbb{Z}}(M) \right)$$

can prove $\left(\begin{array}{c} M' \subseteq M \\ \text{fixed} \end{array}\right)$ $\left\{ \underset{\text{submodule}}{N \subseteq M} \,\middle|\, M' \subseteq N \subseteq M \right\} \underset{\text{preserve order}}{\overset{1:1}{\longleftrightarrow}} \left\{ \begin{array}{c} \text{submodules} \\ \text{of } M/M' \end{array} \right\}$

$$N \longmapsto N/M'$$

$$\text{inverse image of } E \text{ by } M \longrightarrow M/M' \longleftarrow\!\shortmid \; E \subseteq M/M'$$

<u>Example 1.7</u> $M \xrightarrow{f} N$ homomorphism in $\mathrm{Mod}_A$.

kernel of $f := \ker f := \{ m \in M \mid f(m) = 0 \}$, which is a submodule of $M$.

$\left( m \in \ker(f) \Rightarrow am \in \ker f \Rightarrow \ker(f) \text{ is stable under } A\text{-action} \right)$

image of $f := \mathrm{Im}(f) = f(M)$ submodule of $N$.

cokernel of $f := \mathrm{coker}(f) = N/\mathrm{Im}(f)$ quotient module of $N$.

We have an isom $\mathrm{coim}(f) = M/\ker(f) \cong \mathrm{Im}(f)$ [abelian cat 所具有的性质]

<u>Def/Example 1.8</u> $A$: ring. View $A$ as $A$-module.

An ideal $I$ of $A$ is a submodule $I \subseteq A$, i.e, $I$ is an additive subgroup $I \subseteq A$ such that $A \cdot I \subseteq I$ $(\forall x \in A, \forall y \in I \Rightarrow xy \in I)$.

In this case, $A/I$ is a ring (quotient ring) with a surjective ring homo $\phi : A \twoheadrightarrow A/I$, and we have

$$\left\{ \underset{\text{ideal}}{J \subseteq A} \,\middle|\, I \subseteq J \subseteq A \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{ideals of } A/I \right\}$$

$$J \longmapsto J/I$$

$$\phi^{-1}(\bar{J}) \longleftarrow\!\shortmid \; \bar{J} \subseteq A/I$$

$$\phi : A \twoheadrightarrow A/I$$

Any ring hom $f: A \longrightarrow B$ of rings induces: (view B as A-module)

— $\operatorname{Ker} f = f^{-1}(0) \subseteq A$ ideal of A

— $\operatorname{Im}(f) = f(A) \subseteq B$ | A-submodule of B
subring, but may not be an ideal of B (not B-
e.g. $\mathbb{Q} \xrightarrow{f} \mathbb{Q}[X]$, $f(\mathbb{Q}) \subseteq \mathbb{Q}[X]$ containing unit

— $A/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$.

## 1.9 Operations on submodules/ideals

Localization / Fractions/tensor product 以后讲.

(1) $M \in \mathrm{Mod}_A$, $M_i \subseteq M (i \in I)$ submodules.

sum $\sum M_i := \left\{ \sum x_i \;\middle|\; \begin{array}{l} x_i \in M_i \\ x_i = 0 \text{ for almost all } i \end{array} \right\}$ is the smallest submodule of M containing all $M_i$

intersection $\bigcap M_i \subseteq M$ is still a submodule.

Can show • $\dfrac{M_1 + M_2}{M_1} \cong \dfrac{M_2}{M_1 \cap M_2}$ for all $M_1, M_2 \subseteq M$ submodules

$\left(\operatorname{Ker}\left(M_2 \longrightarrow M_1 + M_2 \longrightarrow \dfrac{M_1+M_2}{M_1}\right) = M_1 \cap M_2\right)$

• If $N \subseteq M \subseteq L$ are A-modules, then $\dfrac{L/N}{M/N} \cong L/M$

$\left(\operatorname{Ker}(L/N \longrightarrow L/M) = M/N\right)$

• Same for ideals.

(2) $I \subseteq A$ ideals, $M \in \mathrm{Mod}_A$

$$IM := \underset{\text{错误}}{\overset{I\otimes M \longrightarrow M}{\xcancel{\mathrm{Im}(I\times M \longrightarrow M)}}} = \left\{ \sum_{\text{finite sum}} a_i x_i \;\middle|\; \begin{matrix} a_i \in I \\ x_i \in M \end{matrix} \right\}$$ is a submodule of $M$.

$I, J \subsetneq A$ ideals.

<u>product of $I$ and $J$</u> $= IJ \; (\subseteq I \cap J)$ is the ideal generated by all $xy \; (x \in I, y \in J)$

Similar to define $I_1 I_2 \cdots I_n$ and powers $I^m \; (m>0)$.

Notation: $I^0 = A = (1)$.

$I^m$ is the ideal generated by all products $x_1 \cdots x_n \; (x_i \in I)$

(3) For $m \in M$, define $(m) = Am = \{am \mid a \in A\} \subseteq M$ submodule

$a \in A$, $(a) \subseteq A$ ideal generated by $a$.

If $M = \sum_{i\in I} Am_i$, then we say $\{m_i\}$ is a set of generators of $M$.

An $A$-module $M$ is said to be finitely ~~generated~~ generated if it has a finite set of generators. ($\Leftrightarrow$ $\exists$ surjective homo

$$\begin{tikzcd} A^n \arrow[r, two heads] \arrow[d, equal] & M \\ A \times A \times \cdots \times A & \end{tikzcd}$$

)

(also say $(a_1, \cdots, a_n) \subseteq A$ f.g. ideal)

(4) $(M_i)_{i\in I}$ family of A-modules, can define

direct sum $\bigoplus_{i\in I} M_i = \left\{ (x_i)_{i\in I} \;\middle|\; x_i \in M_i,\ x_i = 0 \text{ for almost all } i\in I \right\}$

direct product $\prod_{i\in I} M_i = \left\{ (x_i)_{i\in I} \;\middle|\; x_i \in M_i \right\} \supseteq \bigoplus_{i\in I} M_i$

when $|I| < \infty$, then $\bigoplus M_i = \prod M_i$.

$A_i$ : ring, $A = \prod_{i=1}^{n} A_i$, $A_j \overset{\text{not preserve unit}}{\hookrightarrow} \prod_{i=1}^{n} A_j$ not subring ([illegible])

$a_j \longmapsto (0, \cdots, 0, a_j, 0, \cdots, 0)$ but an ideal

and $\prod_{i=1}^{n} A_i = \bigoplus_{j=1}^{n} A_j$ as A-modules.

Conversely, given a (module) decomposition $A = I_1 \oplus \cdots \oplus I_n$ of $A$ as a direct sum of ideals/modules, we have

$$A \cong \prod_{i=1}^{n} A/J_i, \quad J_i = \bigoplus_{j\neq i} I_j .$$

Each $I_i$ is a ring (isomorphic to $A/J_i$), ideal

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$I_i \subseteq A$ submodule with surjection $A \twoheadrightarrow I_i$

$\mathrm{Ker}(A \twoheadrightarrow I_i) = \bigoplus_{j\neq i} I_j \Rightarrow I_i \cong A / \bigoplus_{j\neq i} I_j$ and $I_i$ has a ring structure.

(5) $N, P \subseteq M$ submodules.

$(N:P) = \left\{ \substack{\text{形式上的} \\ a \in \frac{N}{P}} \right\} = \{ a \in A \mid a \cdot P \subseteq N \}$ is an ideal of $A$.

for example: $(0:M) = \{ a \in A \mid aM = 0 \}$ = annihilator of $M$
= $\mathrm{Ann}(M)$

can show: $(N:P) = \mathrm{ann}\left(\frac{N+P}{N}\right)$

For ideals $I, J \subseteq A$, same define $(I:J) = \{ x \in A \mid xJ \subseteq I \}$

$\mathrm{ann}(I) = (0:I) = \{ x \in A \mid xI = 0 \}$ annihilator of $I$

Example $\bigcup\limits_{\substack{x \neq 0 \\ \text{in } A}} \mathrm{ann}(x) = \left\{ \substack{\text{zero} \\ \text{divisors} \\ \text{in } A} \right\} =: D$ (书中记号)

$x \in A$ is a zero divisor if $\exists y \neq 0$ s.t $xy = 0$.

For $M \in \mathrm{Mod}_A$, and any ideal $\mathfrak{a} \subseteq \mathrm{Ann}(M)$, $M$ can be regarded as an $A/\mathfrak{a}$-module since $(x + \mathfrak{a}) . \underset{\substack{\in \\ M}}{m} = x m$

也即: $M \in \mathrm{Mod}_A \Rightarrow M \in \mathrm{Mod}_{A/\mathrm{Ann}(M)}$

For any $A$-module $M$, $M$ is a faithful $A/\mathrm{ann}(M)$-module
(ann is zero)

One can show $\mathrm{Ann}(M+N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$

$(N:P) = \{ a \in A \mid aP \subseteq N \} = \{ a \in A \mid a \cdot \frac{N+P}{N} = 0 \}$
$= \mathrm{ann}\left(\frac{N+P}{N}\right)$

**Exercise 1.11** (1) $I \subseteq (I:J)$ since $I \cdot J \subseteq I$

(2) $(I:J)J \subseteq I$ by def.

(3) $\big((I:J):K\big) = (I:JK) = \big((I:K):J\big)$

$\parallel$ $\mathrm{ann}\cancel{\left(\frac{(I:J)+K}{(I:J)}\right)}$

(4) $\left(\bigcap_i I_i : J\right) = \bigcap_i (I_i : J)$

(5) $\left(I : \sum J_i\right) = \bigcap_i (I : J_i)$.

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2024.03.06 (更正：$IM \neq \mathrm{Im}(I \times M \to M)$，为 $\mathrm{Im}(I \otimes M \to M)$

Now we turn to prime ideals and maximal ideals.

**Def 1.12** Recall that integral domain = ring with no zero divisor
$(\Rightarrow 1 \neq 0)$

Field = ring in which $1 \neq 0$ and every non-zero element is a unit.

$\mathfrak{p} \subseteq A$ ideal

$\mathfrak{p}$ is a prime ideal $\Leftrightarrow \left(\begin{array}{l}\mathfrak{p} \neq (1) \\ \forall\, xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}\end{array}\right)$

$\Leftrightarrow A/\mathfrak{p}$ is an integral domain

$\Big( A/\mathfrak{p} \neq 0$ 且 $\bar{x} \cdot \bar{y} = 0 \ (xy \in \mathfrak{p})$

$\Rightarrow \bar{x} = 0$ or $\bar{y} = 0$

i.e., $x \in \mathfrak{p}$ or $y \in \mathfrak{p} \Big)$

$\mathfrak{p}$ is a maximal ideal $\Leftrightarrow$ $\mathfrak{p} \neq (1)$ and there is no ideal $I$ such that $\mathfrak{p} \subsetneq I \subsetneq (1)$

$\Leftrightarrow$ The only ideal in $A/\mathfrak{p}$ is $0$ and $(1)$.

$\Leftrightarrow$ $A/\mathfrak{p}$ is a field.

In particular, maximal ideals are prime ideals.

$A$ is an integral domain $\Leftrightarrow$ $A/(0)$ is an integral domain

$\Leftrightarrow$ $(0)$ is a prime ideal.

<u>Definition 1.13</u> $\operatorname{Spec} A = \left\{ \begin{matrix} \text{prime ideals} \\ \text{of } A \end{matrix} \right\}$

<u>Zariski topology on Spec A</u>

(1) For $M \subseteq A$, closed subset $V(M) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid M \subseteq \mathfrak{p} \}$

closed sets in Spec A are subsets of the form $V(M)$.

(2) If $f \in A$, $D(f) := \operatorname{Spec} A \setminus V(f)$ (elementary open subset)

they form a basis of open sets of the Zariski topology of Spec A

习题：它们确实定义了一个拓扑.

$f \in \mathfrak{p} \Leftrightarrow \mathfrak{p} \in V(f) = \left\{ \begin{matrix} \text{vanishing locus} \\ \text{of } f \end{matrix} \right\} = \{ \mathfrak{p} \in \operatorname{Spec} A \mid \underset{\uparrow}{f \in \mathfrak{p}} \}$

f vanishes at the point $\mathfrak{p} \in \operatorname{Spec} A$

$f \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \notin V(f) \Leftrightarrow \mathfrak{p} \in D(f)$.

<u>Basic example</u> $k$ = alg. closed. $f \in k[x]$ polynomial.

$\alpha \in k$ $(x-\alpha) \in \operatorname{Spec} k[x]$,

$f(\alpha) = 0 \Leftrightarrow f \in (x-\alpha) \Leftrightarrow$ ~~$\mathfrak{p} \in V((x-\alpha))$~~ $(x-\alpha) \in V(f) \overset{1:1}{=} \operatorname{Spec} A/f$

Thm(Hochster) For a Topo. space $X$, the following assertions are equivalent:

(1) $\exists$ ring $A$, $X \cong \operatorname{Spec} A$

(2) One can write $X$ as inverse limit of finite $T_0$-space.

(3) $X$ is spectral, i.e., $X$ is quasi-compact, has a basis of quasi-compact open subsets stable under finite intersections, and every irr. closed subs has a unique generic point (sober)

Construction 1.14 Any ring homomorphism $f: A \to B$ defines a map

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "{}^a f"] & \operatorname{Spec} A \\ \underset{\text{prime of } B}{q} \arrow[r, mapsto] & f^{-1}(q) \end{tikzcd}$$

(*) which is again a prime

$f^{-1}(q)$ !! $p := q \cap A$

$\Rightarrow$ Note that $A/f^{-1}(q) \longrightarrow B/q$ injective homomorphism and $B/q$ is integ

$\Rightarrow A/f^{-1}(q)$ is integral domain

$\Rightarrow f^{-1}(q)$ is a prime ideal in $A$.

$\boxed{\text{In case } (*)\text{, we say } q \text{ lies over } p = q \cap A.}$

Question 1.15 If $m \subseteq B$ is a maximal ideal, does $f^{-1}(m)$ maxima

$$\underset{\text{field}}{B/m} \hookleftarrow \underset{\text{may not be a field.}}{A/f^{-1}(m)}$$

e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$. (0) maximal in $\mathbb{Q}$ but not maximal in $\mathbb{Z}$.

prime ideal 是交换代数中最 fundamental 的定义，对应于 Spec A 中点.（不一定闭）

接下来，讨论若干断言 prime ideal 存在性 或构造 prime ideal 的命题.

**Thm 1.16** Every commutative ring $A \neq 0$ has at least one maximal ideal.

(Spec A 中有闭点)

We apply **Zorn's Lemma**: Let $S \neq \phi$ be a partially ordered set
($\exists$ relation $x \leq y$ on $S$ which is reflexive, transitive such that $(x \leq y) \wedge (y \leq x) \Rightarrow y = x$)
A chain $T \subseteq S$ is a subset such that for all $x, y \in T$ either $x \leq y$ or $y \leq x$.

If every chain $T$ of $S$ has an upper bound in $S$, then $S$ has at least one maximal element.

**proof of 1.16** define $\Sigma = \{ I \mid \underset{\text{ideal}}{I} \subsetneq (1) = A \}$, ordered by inclusion.

Since $(0) \in \Sigma \Rightarrow \Sigma$ is non-empty.

We show: Every chain $(I_\alpha) \subseteq \Sigma$ has an upper bound in $\Sigma$

$(\forall \alpha, \beta, I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha)$

Indeed, $I = \bigcup_\alpha I_\alpha$ is an upper bound $(1 \notin I \Rightarrow I \in \Sigma)$

By Zorn's lemma $\Rightarrow \Sigma$ has a maximal element. ▨

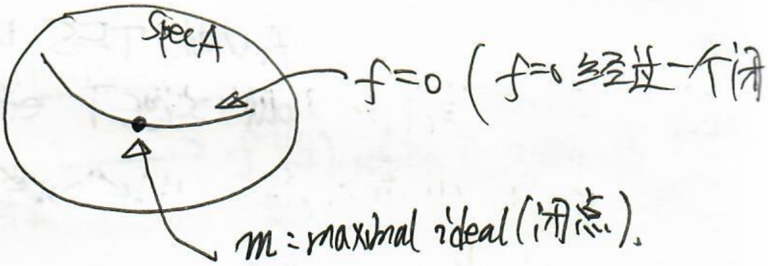
**Remark 1.17** 在1.16中，若假设A是Noetherian，则不用 Zorn引理.

Corollary 1.18 If $I \neq (1)$ is an ideal of $A$, then there is a maximal ideal of $A$ containing $I$.

proof Apply 1.16 to $A/I$. □

Corollary 1.19 Every non-unit of $A$ is contained in a maximal ideal.

Pf. If $f$ is non-unit, then $I = (f) \neq (1)$, and apply 1.18. ▦



Definition 1.20 If the ring $A$ has exactly one maximal ideal $\underline{m}$, th

we call $A$ a local ring.

We call $k = A/m$ the residue field of $A$.

( Localization at a point $\mathfrak{p}$ $\rightsquigarrow$ produce a local ring $\underset{\overset{\|}{(A \setminus \mathfrak{p})^{-1}A.}}{A_{\mathfrak{p}}}$ with maxima ideal $\mathfrak{p}A_{\mathfrak{p}}$, residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} =$

$= \mathrm{Frac}($

比如 $\mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{F}_p$.

$\mathbb{Z}_{(p)} = \{\frac{r}{s} \mid p \nmid s\}$, $p\mathbb{Z}_{(p)}$, $\mathbb{F}_p$.

$k((T))$ = field of formal power series

$k[[T]]$, $(T)$, ~~$(k)$~~ $k$.

$S$ : Riemann surface, i.e., a 1-dim complex manifold.

$p \in S$.

The ring $R_p \subseteq \mathbb{C}[[T]]$ of functions holomorphic in any neigh. of $P$ is a local ring (DVR) with residue field $\mathbb{C}$.

$R_p$ is the subring of convergent power series in $\mathbb{C}[[T]]$.

**Prop 1.21** (1) If $\mathfrak{m} \subsetneq A$ is a maximal ideal such that every $x \in A \setminus \mathfrak{m}$ is a unit in $A$, then $A$ is a local ring with maximal ideal $\mathfrak{m}$.

**proof** Every ideal $\neq (1)$ $\overset{\text{只含 non-units}}{\text{consists of}}$ non-units, hence is contained in $\mathfrak{m}$

$\Rightarrow \mathfrak{m}$ is the only maximal ideal of $A$. ▦

(2) $\mathfrak{m} \subsetneq A$ maximal ideal such that every element of $1+\mathfrak{m}$ is a unit in $A$. Then $A$ is a local ring.

**proof** We show $\underset{\text{elements in}}{\wedge}$ $A-\mathfrak{m}$ are units (hence the conclusion by (1)).

For $x \in A-\mathfrak{m}$, as $\mathfrak{m}$ maximal $\Rightarrow \mathfrak{m}+(x) \supsetneq \mathfrak{m}$

$\Rightarrow \mathfrak{m}+(x) = (1) = A$

$\Rightarrow \exists y \in A,\ t \in \mathfrak{m},\ \underset{\mathfrak{m}}{\underset{\wedge}{t}} + \underset{\text{in} A}{\underset{\uparrow}{y}}x = 1.$ $\Rightarrow xy = 1 - t \in 1+\mathfrak{m}$ is a unit

$\Rightarrow x$ is a unit ▦

## Example 1.22

(1) $k$ field, $A = k[x_1, \dots, x_n]$, $f \in A$ irreducible polynomial.

By unique factorization, the ideal $(f)$ is a prime,

Indeed, if $gh \in (f) \Rightarrow f \mid gh \Rightarrow f \mid g$ or $f \mid h$. ▦

(2) Every ideal in $\mathbb{Z}$ is of the form $(m)$, $m \geq 0$.

$(m)$ is a prime $\Leftrightarrow$ $m = 0$ or $m$ is a prime number.

In this case, $\mathbb{Z}/m$ is a field, thus $(m)$ is also maximal ide

(3) In $k[x]$, for irreducible polynomial $f \in k[x]$, $(f)$ is maximal.

But for $n > 1$, $(f) \subsetneqq k[x_1, \dots, x_n]$ may not be maximal, for example, take $f = X$

More generally, consider the maximal ideal $\mathfrak{m} = \ker\Big( k[X_1, \dots, X_n] \twoheadrightarrow$
$f \longmapsto f($

$$\mathfrak{m} = \left\{ f \in k[x_1, \dots, x_n] \;\middle|\; \begin{matrix} f \text{ has} \\ \text{zero constant term} \end{matrix} \right\} = (X_1, \dots, X_n).$$

For $n > 1$, $\mathfrak{m}$ is not a principal ideal (in fact, it requires at least $n$ generators).

(4) principal ideal domain $:=$ integral domain in which every ideal is principal.

In such a ring, every non-zero prime ideal is maximal.

Indeed, if $(x) \neq 0$ is a prime ideal, and if $(x) \subsetneqq (y)$, we show $(y) =$

Indeed, since $(x) \subsetneq (y)$ $\Rightarrow x = yz$ in $(y)$

but $x = yz \in (x)$ prime ideal

$\Rightarrow z \in (x)$ $\left(\text{or } y \in (x)\ .\ \text{(not the case)}\right)$

$\Rightarrow z = tx \Rightarrow x = yz = ytx$

domain $\Rightarrow 1 = yt \Rightarrow (y) = (1)$. #

(5) <u>Ideal的几何来源</u>

k field, $Y \subseteq k^n$ subset, $I(Y) = \left\{ f \in k[X_1, \cdots, X_n] \;\middle|\; \begin{array}{l} f(P) = 0 \\ \forall\ P \in Y \end{array} \right\}$

$I(Y)$ is an ideal of $k[X_1, \cdots, X_n]$.

反之, given ideal $I \subseteq k[X_1, \cdots, X_n]$, put $Z(I) = \left\{ \begin{array}{c} \text{zeros of} \\ \text{polynomials in } I \end{array} \right\}$

$Z(I) \subseteq k^n$ subset.

是否1:1对应? ×

$I(Y)$ 何时为 prime ideal?

$\mathrm{Spec}A \ni \mathfrak{P}_0$, $\overline{\{\mathfrak{P}_0\}} \ni \mathfrak{P}_1 \Rightarrow \mathfrak{P}_0 \subseteq \mathfrak{P}_1$

Consider $\overline{\{\mathfrak{P}_1\}} \ni \mathfrak{P}_2$ - - - until get a closed point.

上述问题与以下定义相关:

$$\begin{array}{ccc} \text{nilradical} & \subseteq & \text{Jacobson radical} \\ \| & & \| \\ \bigcap \underline{\text{prime}} & & \bigcap \underline{\text{maximal ideal}} \end{array}$$

Prop 1.23 (1) The set $\mathfrak{n} = \left\{ a \,\middle|\, \begin{array}{l} a \text{ is nilpotent} \\ \text{i.e., } \exists n \text{ s.t. } a^n = 0 \end{array} \right\}$ is an ideal, and

$A/\mathfrak{n}$ has no nilpotent element $\neq 0$.

The ideal $\mathfrak{n}$ is called the nilradical of $A$.

(2) $\mathfrak{n} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

(3) For Jacobson radical $R = \bigcap_{\mathfrak{m} \subseteq A \text{ maximal}} \mathfrak{m}$, we have

$x \in R \Leftrightarrow 1 - xy$ is a unit in $A$ for all $y \in A$

proof (1) $\mathfrak{n}$ is an ideal

If $x \in \mathfrak{n}$, then clearly $Ax \subseteq \mathfrak{n}$.

If $x, y \in \mathfrak{n}$, then $x + y \in \mathfrak{n}$ is nilpotent

$x^m = 0$, $y^n = 0$

$$(x+y)^{m+n-1} = \sum_{r+s=m+n-1} x^r y^s \xrightarrow[x^r y^s = 0]{r \geq m \text{ or } s \geq n}$$

$\Rightarrow \mathfrak{n}$ is an ideal.

We show $A/\mathfrak{n}$ has no nilpotent element $\neq 0$.

For $x \in A$, if $\bar{x} \in A/\mathfrak{n}$ is nilpotent such that $\bar{x}^n = 0$

then $x^n \in \mathfrak{n} \Rightarrow \exists k > 0$ st $(x^n)^k = 0 \Rightarrow x \in \mathfrak{n}$

$\Rightarrow \bar{x} = 0$.

(2) We show $\mathfrak{n} = \mathfrak{n}' := \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{n}$.

We first show $\mathfrak{n} \subseteq \mathfrak{n}'$: $\forall f \in \mathfrak{n}$ st $f^r = 0$

$\Rightarrow f^r \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$

$\Rightarrow f \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime $\Rightarrow f \in \mathfrak{n}' = \bigcap_{\mathfrak{p}} \mathfrak{p} \Rightarrow \mathfrak{n} \subseteq \mathfrak{n}'$.

Now we show $\mathfrak{n}' \subseteq \mathfrak{n} \left( \Leftrightarrow A - \mathfrak{n} \subseteq A - \mathfrak{n}' = \bigcup_{\mathfrak{p} \in \operatorname{Spec} A} A \setminus \mathfrak{p} \right)$

Lemma Suppose $f \in A - \mathfrak{n}$ is not nilpotent, then there is a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$. $(\forall n>0, f^n \notin \mathfrak{p})$.

[This implies $A \setminus \mathfrak{n} \subseteq A \setminus \mathfrak{n}'$]

proof of Lemma Let $\Sigma = \left\{ \underset{\text{ideal}}{\mathfrak{a} \subseteq A} \;\middle|\; \begin{matrix} \forall n>0 \\ f^n \notin \mathfrak{a} \end{matrix} \right\}$ ordered by inclusion.

since $0 \in \Sigma \Rightarrow \Sigma$ is not empty ($\Sigma$中chain有 upper bound)

Zorn's lemma $\Rightarrow \Sigma$ has a maximal element.

Let $\mathfrak{p} \in \Sigma$ be a maximal element. $xy \notin \mathfrak{p} \Rightarrow \mathfrak{p} \subsetneq \mathfrak{p} + (xy)$

We show $\mathfrak{p}$ is a prime ideal

Let $x, y \notin \mathfrak{p}$, we show $xy \notin \mathfrak{p}$ (只要证明 $\mathfrak{p} + (xy) \notin \Sigma$)

$x, y \notin \mathfrak{p} \Rightarrow \mathfrak{p} \subsetneq (\mathfrak{p} + (x) \notin \Sigma$

$\mathfrak{p} \subsetneq \mathfrak{p} + (y) \notin \Sigma$ since $\mathfrak{p}$ maxi

$\Rightarrow \exists \, {}^{m}_{n}$ s.t. $\left. \begin{matrix} f^m \in (\mathfrak{p} + (x)) \\ f^n \in \mathfrak{p} + (y) \end{matrix} \right\} \Rightarrow f^{m+n} \in \mathfrak{p} + (yx)$

$\Rightarrow \mathfrak{p} + (xy) \notin \Sigma$ by def of $\Sigma$

$\Rightarrow (xy) \notin \mathfrak{p} \Rightarrow \mathfrak{p}$ is a prime ideal.

proof of ③ $R = \bigcap_{m:\max} m$. Show $R = \{x \in A \mid 1-xy \text{ is a unit in } A \text{ for all } y \in A\}$

"$\Rightarrow$" Suppose $x \in R = \bigcap_{m:\max} m$. If $\exists y$ s.t. $1-xy$ is not a unit,

then by Corollary 1.19 $\Rightarrow \exists$ maximal ideal $m \ni 1-xy$.

But $x \in m \Rightarrow 1 \in m$ a contradiction!

"$\Leftarrow$" Suppose $1-xy$ is a unit in A for all $y \in A$.

We show $x \in \bigcap_{m:\text{maxi}} m$

If not, $\exists$ maximal $m$ s.t. $x \notin m \Rightarrow m + (x) = (1) = A$.

$\Rightarrow u + xy = 1$ for some $u \in m$, $y \in A$

$\Rightarrow 1 - xy \in m$ and is therefore not a unit, 与矛

## 总结一下上面的证明

Recall that A subset $S \subseteq A$ ring is called a multiplicative subset of A if $1 \in S$ and if the product of elements in S are agin

$S \subseteq A$ multiplicative subset such that $0 \notin S$ (S中元素不是nilpot

$$\Sigma = \{I \underset{\text{ideal}}{\subseteq} A \mid I \cap S = \phi\}.$$

Since $0 \in \Sigma \Rightarrow \Sigma$ is non-empty.

By Zorn's lemma $\Rightarrow \Sigma$ has a maximal element $\mathfrak{p} \in \Sigma$ such that $\mathfrak{p}$ must be a prime (同证明 1.15).

Thus we proved: (以后用分式构造 $S^{-1}A$ 重记).

Corollary 1.24 If $0 \notin S$ is a multiplicative subset of a ring $A$, then there is a prime $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \cap S = \phi$.

nilradical $= \{a \in A \mid a \text{ is nilpotent}\}$ 的推广.

Definition 1.25 $I \subseteq A$ ideal.

$$(\text{radical of } I) = \sqrt{I} = r(I) := \left\{ x \in A \,\middle|\, \begin{matrix} \exists n \geq 1 \text{ s.t.} \\ x^n \in I \end{matrix} \right\}$$

$$= \{ x \in A \mid \bar{x} \text{ is nilpotent in } A/I \}$$

$$A \xrightarrow{\phi} A/I \qquad \phi^{-1}\Big( \bigcap_{\substack{\bar{\mathfrak{p}} \in \operatorname{Spec} A \\ I \subseteq \mathfrak{p}}} \bar{\mathfrak{p}} \Big) = \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p}: \text{prime}}} \mathfrak{p}$$

特别, $$\sqrt{0} = \sqrt{(0)} = \left\{ x \in A \,\middle|\, \begin{matrix} \exists n \geq 1 \text{ s.t.} \\ x^n = 0 \end{matrix} \right\} = \text{nilradical of } A.$$

总结:

Prop 1.26 $I \subseteq A$ ideal. The radical $\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}$.

Exercise 1.27 ① $I \subseteq \sqrt{I} = \sqrt{\sqrt{I}} = \cdots$

② $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

$$\underset{\displaystyle\bigcap_{IJ \subseteq \mathfrak{p}} \mathfrak{p}}{\sqrt{IJ}} \qquad \qquad \underset{\displaystyle\bigcap_{I \subseteq \mathfrak{p}} \mathfrak{p}}{\sqrt{I}}$$

$IJ \subseteq \mathfrak{p}$ $\xleftrightarrow{\text{参见 Prop 1.33}}$ $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$

③ $\sqrt{I} = (1) \Leftrightarrow I = (1)$.

④ $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

⑤ If $\mathfrak{p}$ is a prime, then $\sqrt{\mathfrak{p}^n} = \sqrt{\mathfrak{p}} = \mathfrak{p}$ for all n≥

⑥ For any $\underset{\text{subset}}{E \subseteq A}$, can still define $\sqrt{E}$, but $\sqrt{E}$ may not be an ideal. We have $\sqrt{\bigcup_{\alpha} E_\alpha} = \bigcup_{\alpha} \sqrt{E_\alpha}$.

问题: For ideal $I \subseteq A$, ~~which~~ what is $\displaystyle\bigcap_{\substack{I \subseteq m \\ m:\text{maximal}}} m$.

Source of radicial ideals

$I \subseteq A$ ideal $\Rightarrow$ closed subset $V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec} A$

$$V(I) = V(\sqrt{I})$$

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p}} \mathfrak{p} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

2024.03.11

**Prop 1.28** Let $D = \left\{ \begin{matrix} \text{zero divisors} \\ \text{of } A \end{matrix} \right\} = \bigcup_{x \neq 0} \mathrm{Ann}(x)$.

Then $D = \bigcup_{x \neq 0} \sqrt{\mathrm{Ann}(x)}$.

**proof** $D = \sqrt{D}$ ( $D \subseteq \sqrt{D}$ by def.
$\sqrt{D} \subseteq D$ since if $f \in \sqrt{D}$, $\Rightarrow f^n$ ~~$x$~~ is a zero divisor $\Rightarrow f$ is a zero divisor )

$$= \sqrt{\bigcup_{x \neq 0} \mathrm{Ann}(x)} = \bigcup_{x \neq 0} \sqrt{\mathrm{Ann}(x)}$$

**Example 1.29** If $A = \mathbb{Z}$, $I = (m)$, $m = p_1^{r_1} \cdot \ldots \cdot p_t^{r_t}$ $(r_i \geq 1)$

Then $\sqrt{(m)} = (p_1 \cdot \ldots \cdot p_t) = \bigcap_{i=1}^{r} (p_i)$

$\uparrow$ coprime ideal 的性质 (待会证).

**Definition 1.30** Two ideals $I$ and $J$ are said to be coprime (or comaximal) if $I + J = (1)$ ($\Leftrightarrow \exists x \in I, y \in J$, s.t. $x + y = 1$)

In this case, $I \cap J = I \cdot J$:

$IJ \subseteq I \cap J$ easy by def of ideals

$$I \cap J = (I + J)(I \cap J) = I(I \cap J) + J(I \cap J) \subseteq IJ + IJ \subseteq IJ$$

更一般地, 有

**Prop 1.31** $A$ = ring, $I_1, \ldots, I_n$ : ideals of $A$, define a homomorphism

$$\phi : A \longrightarrow \prod_{i=1}^{n} A/I_i \quad x \longmapsto (x + I_1, \ldots, x + I_n)$$

(1) If $I_i$ and $I_j$ are coprime for $i \neq j$, then $\prod I_i = \bigcap I_i$

Pf Induction on $n$

$n=2$ true by 1.30.

Suppose $n>2$, then $J := \prod_{n=1}^{n-1} I_i = \bigcap_{i=1}^{n-1} I_i$ (induction step)

Since $I_i + I_n = (1) \left(\forall 1 \leq i \leq n-1\right) \Rightarrow \exists x_i \in I_i, y_i \in I_n$ s.t $x_i + y_i = 1$

$$\Rightarrow \prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1}(1-y_i) \equiv 1 \mod I_n$$

$$\Rightarrow I_n + J = (1) \Rightarrow \prod_{i=1}^{n} I_i = J I_n = J \cap I_n = \bigcap_{i=1}^{n} I_i$$

(2) $\phi$ is injective $\Leftrightarrow \bigcap I_i = (0)$.

clearly by $\ker\phi = \bigcap I_i$.

(3) 中国剩余定理

$\phi$ is surjective $\Leftrightarrow$ $I_i$ and $I_j$ are coprime whenever $i \neq j$

$\Rightarrow A/\bigcap I_i \cong A/I_1 \times \cdots \times A/I_n$ iff $I_i + I_j = (1)$ for all $i \neq j$.

proof "$\Rightarrow$" Let us show that $I_1$ and $I_2$ are coprime (其它类似)

$\exists x \in A$ s.t $\phi(x) = (1, 0, \cdots, 0)$, hence $x \equiv 1 \mod I_1$

$x \equiv 0 \mod I_2$

$\Rightarrow 1 = (1-x) + x \in I_1 + I_2 \Rightarrow I_1 + I_2 = (1)$.

"$\Leftarrow$" enough to show: $\exists x \in A$ s.t $\phi(x) = (1, 0, \cdots, 0)$ [其它类似].

Since $I_1 + I_i = (1)$ for $i > 1$, we have $u_i + v_i = 1$ $(u_i \in I_1, v_i \in$

take $x = \prod_{i=2}^{n} v_i$, then $x = \prod(1-u_i) \equiv 1 \mod I_1$

$x \equiv 0 \mod I_j$

$\Rightarrow \phi(x) = (1, 0, \cdots, 0)$ as required. $\boxplus$

**Exercise 1.32** $I, J \subseteq A$ ideals. If $\sqrt{I}$ and $\sqrt{J}$ are coprime, then $I$ and $J$ are coprime.

pf $\sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}} = \sqrt{(1)} = (1) \Rightarrow I+J=(1)$. $\square$

**Proposition 1.33 (Prime avoidance lemma)**

(1) Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be prime ideals of $A$ and let $I \subseteq A$ be an ideal.

then $I \subseteq \bigcup_{i=1}^{n} \mathfrak{P}_i \Rightarrow I \subseteq \mathfrak{P}_i$ for ~~same~~ some $i$.

(我们会证明：若 $I \nsubseteq \mathfrak{P}_i$ for all $i$, then $\exists x \in I$ such that $x \notin \mathfrak{P}_i$ for all $i$).

(2) Let $I_1, \cdots, I_n$ be ideals of $A$ and $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal.

Then $\bigcap_{i=1}^{n} I_i \subseteq \mathfrak{p} \Rightarrow I_i \subseteq \mathfrak{p}$ for some $i$.

If $\mathfrak{p} = \bigcap_{i=1}^{n} I_i$, then $\mathfrak{p} = I_i$ for some $i$.

~~proof~~ (1) Prove by induction on $n$ in the form

$$I \nsubseteq \mathfrak{P}_i \ (1 \leq i \leq n) \Rightarrow I \nsubseteq \bigcup_{i=1}^{n} \mathfrak{P}_i$$

True for $n=1$ ✓.

( $n=2$ 时，若 $I \nsubseteq \mathfrak{P}_1$ 且 $I \nsubseteq \mathfrak{P}_2$, choose $x, y \in I$, $x \notin \mathfrak{P}_1$, $y \notin \mathfrak{P}_2$
We are done unless $x \in \mathfrak{P}_2$ and $y \in \mathfrak{P}_1$. Then $x+y \notin \mathfrak{P}_1$, $x+y \notin \mathfrak{P}_2$ )

If $n>1$ and if the result is true for $n-1$, then for each $i$,

$$I \nsubseteq \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_{i-1} \cup \mathfrak{P}_{i+1} \cup \cdots \cup \mathfrak{P}_n$$

$$\Rightarrow \exists x_i \in I \text{ s.t } x_i \notin \mathfrak{P}_j \ (\forall j \neq i).$$

If $\exists i$, s.t $x_i \notin \mathfrak{p}_i$, then done (此时 $x_i \notin \bigcup_{i=1}^{n} \mathfrak{p}_i$).

If not, then $x_i \in \mathfrak{p}_i$ for all $i$,

consider $y = \sum_{i=1}^{n} x_1 x_2 \cdots x_{i-1} x_{i+1} x_{i+2} \cdots x_n$

$\Rightarrow y \in I$ and $y \notin \mathfrak{p}_i$ $(\forall\, 1 \leq i \leq n)$

$\Rightarrow I \nsubseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$ $\square$

proof of (2) Suppose that $\mathfrak{p} \nsupseteq I_i$ for all $i$.

Then $\exists x_i \in I_i$, $x_i \notin \mathfrak{p}$ $(1 \leq i \leq n)$

$\Rightarrow x_1 \cdots x_n \in \prod I_i \subseteq \bigcap I_i$

but $x_1 \cdots x_n \notin \mathfrak{p}$ (since $\mathfrak{p}$ is a prime)

hence $\mathfrak{p} \nsupseteq \bigcap I_i$

Finally, if $\mathfrak{p} = \bigcap I_i$, then $\mathfrak{p} \subseteq I_i$ $(\forall i)$.

Hence $\mathfrak{p} = I_i$ for some $i$. $\square$

结论1.34 $I \nsubseteq \mathfrak{p}_i$, and all but two of $\mathfrak{p}_i$ are prime ideals.

Then $\exists x \in I$ s.t $x \notin \mathfrak{p}_i$ for all $i$.

~~Slo~~ Solgan 1.35 In an affine scheme Spec$A$, if a finite number of points $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$ is contained in an open

在1.33(1)中取 $I = (x_1, \cdots, x_n)$

subset (比如 $\bigcup_{i=1}^{n} D(x_i)$), then they are contained in a smaller principal open subset (比如 $D(x)$).

## Definition 1.36 (Extension and contraction)

$A \xrightarrow{f} B$ , $I \subseteq A$, $J \subseteq B$ ideals

$f(I) \subseteq B$ may not be an ideal.

$f^{-1}(J) \subseteq A$ is an ideal (if $J$ is a prime, then $f^{-1}(J)$ is a prime)

$J^c := f^{-1}(J) = J \cap A$ called the contraction of $J$

$I^e := f(I)B$ = ideal of $B$ generated by $f(I)$

called the extension of $I$.

问题：$J^c$保 prime, $I^e$是否保 prime？

If $I \subseteq A$ is a prime, $I^e$ may not be a prime.

For example, $f: \mathbb{Z} \hookrightarrow \mathbb{Q}$, $I = (3)$, $I^e = \mathbb{Q}$ which is not a prime.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \\
f(A) \arrow[ur, hook, "j"'] \arrow[ur, phantom, "\text{injective}" near start, shift right=3]
\end{tikzcd}$$

For surjective part:

$$\left\{\begin{matrix}\text{ideals of } A \\ \text{containing } \ker f = \ker p\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{ideals} \\ \text{of} \\ f(A)\end{matrix}\right\}$$

$$\text{prime} \longleftrightarrow \text{prime}$$

For injective part, the general situation is very complicated.

**Example 1.37** $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{-1}]$ 代数数论.

$p \in \mathbb{Z}$ prime, but $(p)^e = p\mathbb{Z}[\sqrt{-1}]$ may not be a prime.

— $(2)^e = ((1+i)^2)$ square of a prime ideal in $\mathbb{Z}[\sqrt{-1}]$, $i = \sqrt{-1}$.

— If $p \equiv 1 \bmod 4$, $(p)^e$ is the product of two distinct prime ideals

$\Leftrightarrow$ p可分解两个素数乘积.

— If $p \equiv 3 \bmod 4$, then $(p)^e$ is a prime in $\mathbb{Z}[\sqrt{-1}]$.

for example, $(5)^e = (2+i)(2-i)$

**Prop 1.38** $A \xrightarrow{f} B$ ring homo, $I \subseteq A$, $J \subseteq B$ ideals.

(1) $I \subseteq I^{ec} = f^{-1}(IB)$

$J \supseteq J^{ce} = f(f^{-1}(J))B$

(2) $J^c = J^{cec}$ (by (1), $J^c \supseteq J^{cec}$ and $J^c \subseteq (J^c)^{ec}$)

$I^e = I^{ece}$ (by (1), $I^e \subseteq I^{ece}$ and $I^e \supseteq (I^e)^{ce}$)

(3) $C = \left\{ \begin{matrix} \text{contracted} \\ \text{ideals in} \\ A \end{matrix} \right\}$ $E = \left\{ \begin{matrix} \text{extended} \\ \text{ideals in} \\ B \end{matrix} \right\}$

Then • $C = \{ I \subseteq A \mid I^{ec} = I \}$ (证明: if $I = J^c$ for some $J \subseteq B$ then $I^{ec} = J^{cec} = J^c =$ 反之若 $I = I^{ec} \Rightarrow I$ is the contra of $I^e$

• $E = \{ J \subseteq B \mid J^{ce} = J \}$

• $C \xleftrightarrow{1:1} E$ (利用 (2), $\Rightarrow$ 双射)

$I \longmapsto I^e$

$J^c \longleftarrow\!\shortmid J$

• E is closed under sum and product

C is closed under intersection and $\sqrt{\cdots}$.

## 讨论 finitely generated modules

**1.39 (free modules)** A free A-module is one which is isomorphic to $A^{(I)} := \bigoplus_{i\in I} A$. When $|I|=n$, we denote it by $A^n = A\oplus\cdots\oplus A$

$A^0 = 0$ zero module.

**Prop 1.40** $M$ is a f.g. A-module $\Leftrightarrow$ $M$ is isomorphic to a quotient of $A^n$ for some $n>0$.

Pf "$\Rightarrow$" $M = Ax_1+\cdots+Ax_n$. Define $A^n \xrightarrow[\text{surj}]{\phi} M$ by $\phi(a_1,\dots,a_n) = a_1x_1+\cdots+a_nx_n$

$\phi$ surj $\Rightarrow M \cong A^n/\ker\phi$.

"$\Leftarrow$" Assume $\exists$ surj $\phi: A^n \twoheadrightarrow M$. Let $x_i = \phi(0,\dots,\overset{i\text{-th place}}{1},0,\dots,0)$

Then $M$ is generated by $\{x_i\}_{1\le i\le n}$. ▦

**Prop 1.41** $M$: f.g. A-module, $I\subseteq A$ ideal.

$\phi: M\to M$ A-module homo such that $\phi(M)\subseteq IM$.

Then $\phi$ satisfies an equation of the form

$$\phi^n + a_1\phi^{n-1}+\cdots+a_n = 0 \text{ in } \mathrm{End}_A(M), \text{ where } a_i\in I.$$

proof Let $M = Ax_1+\cdots+Ax_n$. $\phi(x_i)\in IM$.

Write $\phi(x_i) = \sum_{j=1}^n a_{ij}x_j$ $(1\le i\le n,\ a_{ij}\in I)$

$\Rightarrow \sum_{j=1}^n(\delta_{ij}\phi - a_{ij})(x_j) = 0$, $\delta_{ij}$ Kronecker delta.

$$\Rightarrow (\delta_{ij}\phi - a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \cdot \begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix} = 0$$

multiplying the left by the adjoint of the matrix $(\delta_{ij}\phi - a_{ij})$

$\Rightarrow \det(\delta_{ij}\phi - a_{ij})$ annihilates each $x_i$ $\Rightarrow \det(\delta_{ij}\phi - a_{ij}) = 0$ in $\mathrm{End}_A M$

展开,得到关于$\phi$的方程.

In particular, take $I=A$, $\mathrm{End}_A(M)$中每个元素都满足一个$n$次多项式.

**Corollary 1.4.2** $M$: f.g. $A$-module. $I\subseteq A$ ideal such that $IM=M$.

Then $\exists\, x\equiv 1 \bmod I$ such that $xM=0$.

proof In prop 1.4.1, take $\phi=id: M\longrightarrow M=IM$, $\phi(M)=M=IM$

then $\phi^n+a_1\phi^{n-1}+\cdots+a_n=0$, where $a_i\in I$, in $\mathrm{End}_A(M)$

$$\begin{array}{c} \phi^n+a_1\phi^{n-1}+\cdots+a_n \\ \parallel \\ 1+a_1+\cdots+a_n\in\mathrm{End}_A(M) \end{array}$$

Let $x=1+a_1+\cdots+a_n\equiv 1 \bmod I$. Then $x=0$ in $\mathrm{End}_A(M)$

i.e, $xM=0$.

**Prop 1.4.3 (Nakayama's Lemma)**

$M$: f.g. $A$ module. $I\subseteq A$ ideal contained in the Jacobson radical $R$ of $A$

Then $IM=M$ implies $M=0$.

proof By 1.4.2, $\exists\, x\equiv 1 \bmod I$ s.t $xM=0$.

But $x\in 1+R$ is a unit by prop 1.2.3.(3). $\Rightarrow M=0$.

**Corollary 1.4.4** $M$: f.g. $A$-module, $N\subseteq M$ submodule

$I\subseteq R$ Jacobson radical.

Then 若 $M=N+IM$, then $M=N$.

proof consider $M/N$. Then $I\cdot M/N=I\cdot\dfrac{N+IM}{N}=\dfrac{IM+N}{N}=$

$\Rightarrow \dfrac{M}{N}=0$.

**Prop 1.45** A: local ring with maximal ideal $\mathfrak{m}$.

$k = A/\mathfrak{m}$ residue field.

$M$: f.g. A-module.

$M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$ $\Rightarrow$ $M/\mathfrak{m}M$ is a $k$-module, which is a f.d. $k$-vector space.

If $\{x_i\}_{1\leq i\leq n} \subseteq M$, whose image in $M/\mathfrak{m}M$ form a basis of $M/\mathfrak{m}M$ as $k$-vector space, then $\{x_i\}$ generate $M$.

**Pf** Let $N = Ax_1 + \cdots + Ax_n \subseteq M$. Then $N \to M \to M/\mathfrak{m}M$ surjective

$\Rightarrow N + \mathfrak{m}M = M$ $\Rightarrow N = M$. ▦

(由于 Jacob radical $= \mathfrak{m}$)

~~2024.03.13~~

## A quick introduction to homological algebra (冗之前 2023年讲义)

**Definition 1.46** A sequence of A-modules and A-homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$$

(一般用 upper index
cohomology index)

is said to be a (cochain) complex if $f_{i+1} \circ f_i = 0$ ($\operatorname{Im} f_i \subseteq \ker f_{i+1}$).

— We say it is exact at $M_i$ if $\ker f_{i+1} =$ ~~Image $f_i$~~ $= \operatorname{Im} f_i$

— ______ exact if it is exact at $M_i$ for all $i$.

In general, we define $H_i(M) := \dfrac{\ker f_{i+1}}{\operatorname{Im} f_i}$ (homology/cohomology at $i$)

In particular, $0 \to M' \xrightarrow{f} M$ exact $\Leftrightarrow$ $f$ injective ($\ker f = \{0\}$)

$M \xrightarrow{g} M'' \to 0$ exact $\Leftrightarrow$ $g$ surjective ($\operatorname{Im} g = \ker 0 = M''$)

$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ exact $\Leftrightarrow$ $f$ injective
(short exact sequence)
$g$ surjective
and $g$ induces an isomorphism
$\operatorname{coker} f = M/f(M') \cong M''$

Any long exact sequence can be split into short exact sequences:

$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$ $\rightsquigarrow$ $N_i = \ker f_{i+1} = \operatorname{Im} f_i$

$0 \to N_i \to M_i \to N_{i+1} \to$

2024年3月13日

**Prop 1.47 (exact test)** In the category $\mathrm{Mod}_A$, we have

(1) $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ exact iff $\forall$ $A$-module $N$, the sequence

$0 \to \operatorname{Hom}(M'', N) \xrightarrow{\bar{v}} \operatorname{Hom}(M, N) \xrightarrow{\bar{u}} \operatorname{Hom}(M', N)$ is exact

(特别：反变函子 $\operatorname{Hom}(-, N)$ : ~~$\mathrm{Mod}_A$~~ $\mathrm{Mod}_A \to \mathrm{Mod}_A$ is left exact )

(2) $0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$ exact iff $\forall$ $A$-module $N$, the sequence

$0 \to \operatorname{Hom}(M, N') \xrightarrow{\bar{u}} \operatorname{Hom}(M, N) \xrightarrow{\bar{v}} \operatorname{Hom}(M, N'')$ exact.

(特别：$\operatorname{Hom}(M, -)$ is left exact)

(证明：$v$ surj, 且 $\operatorname{Im} u = \ker v$)

**proof** 只证明 (1)中 "$\Leftarrow$".

Since $\bar{v}$ is injective for all $N$, take $N = M''/\operatorname{Im} v$ get

$0 \to \operatorname{Hom}(M'', M''/\operatorname{Im} v) \to \operatorname{Hom}(M, M''/\operatorname{Im} v)$ exact

特别 $(M'' \underset{\text{cano}}{\twoheadrightarrow} M''/\operatorname{Im} v) \longmapsto (M \xrightarrow{v} M'' \to M''/\operatorname{Im} v)$

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[rr, bend right, "0"'] & M'' \arrow[r] & M''/\operatorname{Im} v \end{tikzcd}$$

thus $M'' \twoheadrightarrow M''/\operatorname{Im} v$ is zero $\Rightarrow$ $M''/\operatorname{Im} v = 0$ $\Rightarrow$ $v$ surjective

Now we show $\operatorname{Im} u = \ker v$

- Since $\bar{u} \circ \bar{v} = 0$ for all $N \Rightarrow v \circ u \circ f = 0$ for all $M'' \xrightarrow{f} N$.

  Take $f = id_{M''} \Rightarrow v \circ u = 0 \Rightarrow \operatorname{Im} u \subseteq \ker v$.

- Now take $N = M/\operatorname{Im} u$ with projection $M \overset{\phi}{\twoheadrightarrow} N$

  Then $\phi \in \ker \bar{u}$, hence $\exists \psi : M'' \longrightarrow N$ such that $\phi = \psi \circ v$

$$\begin{tikzcd} M'' \arrow[r, "\exists \psi"] & N = M/\operatorname{Im} u \\ M \arrow[u, "v"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

$\Rightarrow \ker v \subseteq \ker \phi = \operatorname{Im} u$

~~$\Rightarrow \operatorname{Im} u$~~

□

thus $\operatorname{Im} u = \ker v$.

**问题：对哪些 M与N，$\operatorname{Hom}(M,-)$与$\operatorname{Hom}(-,N)$ 正合**

<u>Definition 1.48</u> A (covariant) functor $F: \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$ (常取 $\mathrm{Mod}_{\mathbb{Z}} = \mathrm{Ab}$) consisting of the following data:

(1) $\forall$ A-module $M$, a B-module $F(M)$.

(2) $\forall$ A-module homomorphism $M \xrightarrow{g} N$,

a B-module homo $F(M) \xrightarrow{F(g)} F(N)$

such that $F(f \circ g) = F(f) \circ F(g)$ (preserve composition)

$F(id_M) = id_{FM}$ (preserve identity)

Moreover, we say $F$ is an <u>additive functor</u>, if moreover each

$$\operatorname{Hom}_A(M,N) \xrightarrow{F} \operatorname{Hom}_B(FM, FN)$$

is a group homomorphism, i.e., $F(0) = 0$ and $F(g+h) = F(g) + F(h)$.

a contravariant functor (反变函子) from $Mod_A \longrightarrow Mod_B$ is a covariant functor $F: Mod_A^{op} \longrightarrow Mod_B$:

— (1) 同

— (2) $M \longrightarrow N \rightsquigarrow F(N) \xrightarrow{F(g)} F(M)$

$$F(f \circ g) = F(g) \circ F(f)$$

$$F(id_M) = id_{FM}.$$

Example 1.49 For any A-module M and N,

$Hom_A(M, -): Mod_A \longrightarrow Mod_A$ additive functor

$Hom_A(-, N)$ additive contravariant functor.

Definition 1.50 $F: Mod_A \longrightarrow Mod_B$, $G: Mod_A^{op} \longrightarrow Mod_B$ additive functors.

(1) We say F (resp. G) is left exact if for any short exact sequence $0 \to M \to N \to R \to 0$ in $Mod_A$, the sequence $0 \to FM \to FN \to FR$ is exact (resp. $0 \to GR \to GN \to GM$) is exact).

(2) ———————— right exact

———————————— the sequence $FM \to FN \to FR \to$ is exact (resp. $GR \to GN \to GM \to 0$ exact).

(3) We say F (resp. G) is exact if F (resp. G) sends (short) exact sequences to (short) exact sequences.

exact $\Longleftrightarrow$ left exact + right exact.

By Prop 1.47, $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_A(-,N)$ are both left exact.

<u>導出函子</u> For a left exact functor $F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$, we will construct a family of functors $\{R^iF : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B\}_{i=0,1,2,\cdots}$ [F的 i-th 右導出函子]

(cohomological $\delta$-functor)

such that (1) $R^0F = F$

(2) For any short exact sequence $0 \to M \to N \to R \to 0$ of A-modules, we have a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & FM \arrow[r] & FN \arrow[r] & FR \arrow[dll] \\
 & R^1FM \arrow[r] & R^1FN \arrow[r] & R^1FR \arrow[dll] \\
 & R^2FM \arrow[r] & R^2FN \arrow[r] & R^2FR \arrow[dll] \\
 & \cdots & &
\end{tikzcd}$$

当F exact 时, 可取
$R^1F = R^2F = \cdots = 0$

(3) Functorial property (a map between two short exact sequences induces a map between long exact sequences)

(Same for right exact functor $F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$
$\rightsquigarrow$ left derived functors $L_iF : \mathrm{Mod}_A \to \mathrm{Mod}_B$
$0 \to M \to N \to R \to 0$ exact
$\rightsquigarrow \cdots \to L_1FM \to L_1FN \to L_1FR \to FM \to FN \to FR \to 0$ exact.)

例 $\mathrm{Hom}(-,N) : \mathrm{Mod}_A^{op} \longrightarrow \mathrm{Mod}_A$ left exact
$\rightsquigarrow \mathrm{Ext}^i(\cancel{M},N) = R^i\mathrm{Hom}(-,N)$, i-th extension group
$\mathrm{Hom}(M,-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A$ left exact
$\rightsquigarrow \mathrm{Ext}^i(M,-) = R^i\mathrm{Hom}(M,-)$.

Balance property: 
$$\begin{tikzcd} \mathrm{Ext}^i(M,N) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & R^i\mathrm{Hom}(M,-)(N). \\ R^i\mathrm{Hom}(-,N)(M) & \end{tikzcd}$$

对于哪些 N 与 M, $\mathrm{Hom}_A(M,-)$ 与 $\mathrm{Hom}_A(-,N)$ 是 exact 的?

Definition 1.51

(1) If $\mathrm{Hom}_A(M,-)$ is an exact funtor, then we say M is a projective A-m 投射模

(2) If $\mathrm{Hom}_A(-,N)$ is an exact funtor, then we say N is an injective A-mo 内射模

Since $\mathrm{Hom}_A(~~*~~,N)$ always left exact, thus we have

N injective $\Leftrightarrow$ $\mathrm{Hom}_A(-,N)$ exact $\Leftrightarrow$ For any injective homo $0 \to I \to R$

the map $\mathrm{Hom}(R,N) \to \mathrm{Hom}(I,N) \to$ su

i.e.
$$\begin{tikzcd} \forall\ I \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed, "?\ \exists"'] \\ N & \end{tikzcd}$$

(any homo $I \to N$ can be extended $R \to N$)

(可包含更大的

Similarly, M is projective $\Leftrightarrow$ $\mathrm{Hom}(M,-)$ exact

$\Leftrightarrow$ For any surjective homo $X \twoheadrightarrow Y$, the map

$\mathrm{Hom}_A(M,X) \to \mathrm{Hom}_A(M,Y)$ is surjective

$\Leftrightarrow$ For any
$$\begin{tikzcd} X \arrow[r, two heads] & Y \\ & M \arrow[u] \end{tikzcd}$$

we can find a lifting
$$\begin{tikzcd} X \arrow[r, two heads] & Y \\ & M \arrow[u] \arrow[ul, dashed, "?"] \end{tikzcd}$$

(可投射到更大的

## 1.52 Injective 对象具有以下性质

(1) If $0\to X\xrightarrow{f} Y\to Z\to 0$ is a short exact seq of A-modules, and if $X$ is an injective A-module, then we have

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[dl, dashed, "\exists g"] & Z \arrow[r] & 0 \\
& \underset{\text{inj}}{X} & & &
\end{tikzcd}$$

$\Rightarrow 0\to X\to Y\to Z\to 0$ split, i.e., $Y\cong X\oplus Z$.

(2) 若 $0\to X\to Y\to Z\to 0$ exact with $X$ injective A-module, then $Y$ injective iff $Z$ injective.

By (1), $Y\cong X\oplus Z$, $\mathrm{Hom}(-,Y)=\underset{\text{exact}}{\mathrm{Hom}(-,X)}\oplus\mathrm{Hom}(-,Z)$.

## 1.53 projective 对象具有以下性质

(1) If $0\to W\to X\to Y\to 0$ exact with $Y$ projective, then $X\cong W\oplus Y$.

(2) Same condition as (1): $X$ is proj $\Longleftrightarrow$ $W$ projective.

**Proposition 1.54** (1) Free A-modules are projective.

(2) $M\in\mathrm{Mod}_A$. $M$ is projective iff $M$ is a direct summand of a free A-mod.

**proof** (1) $\mathrm{Hom}_A(A^{(I)},N)=\prod_I\mathrm{Hom}_A(A,N)=\prod_I N$.

(2) "$\Leftarrow$" $M\oplus N=A^{(I)}\rightsquigarrow\mathrm{Hom}(M,-)$ exact.

"$\Rightarrow$" $M$ proj. Consider $F(M)=\bigoplus_{m\in M}A\xrightarrow{f}\!\!\!\rightarrow M$ surj

$(a_m)_{m\in M}\mapsto\sum a_m\cdot m$

$0\to\ker f\to F(M)\to\underset{\text{proj}}{M}\to 0$. By 1.53, $F(M)=\ker f\oplus M$.

Proposition 1.55 (Baer) A: ring, M: A-module.

M injective $\Leftrightarrow$ For any ideal $I \subseteq A$, any homomorphism of A-modules $I \to$
can be extended to $A \twoheadrightarrow M$.

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & A \arrow[dl, dashed, "\exists"] \\ M & \end{tikzcd}$$

If A is a principal ideal domain (e.g $A = \mathbb{Z}$), then

M inj $\Leftrightarrow$ $\forall f \in A$, any diagram

$$\begin{tikzcd} f \arrow[d, mapsto] & (f) \arrow[r, hook] \arrow[d] & A \arrow[dl, dashed, "\exists"] & 1 \arrow[d, mapsto] & f \arrow[d, mapsto] \\ m \arrow[rrrr, equal, bend right] & M & & n & fn \end{tikzcd}$$

has a solutio

$\Leftrightarrow$ $\forall f \in A \setminus \{0\}$, $\forall m \in M$,

方程 $m = f\underline{x}$ 在 M 中有解 ( "$\frac{m}{f}$" 有意义 )

proof "$\Rightarrow$" clear by def.

$\Leftarrow$ For any inj homo of A-modules $0 \to X \to Y$ and any $\alpha : X \to M$ in $\mathrm{Mod}_A$, we need to find an extension $Y \to M$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\alpha"] & Y \\ & M & \end{tikzcd}$$

$$\Sigma = \left\{ X' \subseteq Y \;\middle|\; \begin{array}{l} X \subseteq X' \subseteq Y \text{ submodules} \\ \text{s.t } \alpha \text{ extends to } X' \xrightarrow{\alpha'} M \end{array} \right\},$$ ordered by inclusion.

$X \in \Sigma \Rightarrow \Sigma$ non-empty.

By Zorn's lemma, $\Sigma$ has a maximal element $X'$

$$\begin{tikzcd} \alpha' : X' \arrow[r] & M \\ X \arrow[u, hook] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

We only need to show $X' = Y$

If $X' \neq Y$. choose $b \in Y \setminus X'$. We will extend $\alpha' : X' \longrightarrow M$ to $X' + Ab$.

$X' \subsetneq Y$

Put $I = \{a \in A \mid ab \in X'\} = (X' : b)$ is an ideal.

$$\begin{tikzcd} a \arrow[d, mapsto] & I \arrow[r, hook] \arrow[d] & A \arrow[ddl, dashed] \\ ab & X' \arrow[d, "\alpha'"'] & \\ & M & \end{tikzcd}$$

has a solution

$\exists f$ s.t $f(a) = \alpha'(ab)$.

Now put $X'' = X' + Ab \subseteq Y$

$$\begin{tikzcd} X' \arrow[d, "\alpha"'] & \subsetneq X'' \subseteq Y \\ M & \end{tikzcd}$$

define $\alpha'' : X'' \longrightarrow M$ by $\alpha''(x' + ab) = \alpha'(x') + f(a)$
$x' \in X', a \in A$.

Check: $\alpha''$ is well-defined If $ab \in X' \cap Ab \Rightarrow \alpha'(ab) = f(a)$ ✓.

Now $\alpha''$ is an extension of $\alpha'$, 与$X'$极大矛盾! ∎

---

$M$: injective $A$-module. $a \in A \setminus \{0\}$, $m \in M$

In $M$, "$\frac{m}{a}$" make sense:

$$\begin{tikzcd} a \arrow[d, mapsto] & (a) \arrow[r, hook] \arrow[d] & A \\ m & M & \end{tikzcd}$$

可除性质.

---

例1.56 $Mod_{\mathbb{Z}} = Ab$, $\mathbb{Q}$ injective $\mathbb{Z}$-module.

$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module

where $\mathbb{Z}_{p^\infty} = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ (also injective).

<u>例 1.57</u> $A$: ring, $I$: injective $\mathbb{Z}$-module.

Then $\mathrm{Hom}_{Ab}(A, I)$ is an injective $A$-module.

<u>Indeed</u>

$$\mathrm{Hom}_A(M, \mathrm{Hom}_{Ab}(A,I)) \xrightarrow{\simeq} \mathrm{Hom}_{Ab}(M, I) \text{ exact in } M.$$

$$\begin{pmatrix} M \to \mathrm{Hom}_{Ab}(A,I) \\ m \mapsto \begin{pmatrix} A \to I \\ a \mapsto f(am) \end{pmatrix} \end{pmatrix} \longleftarrow\!\!\dashv \quad (M \xrightarrow{f} I)$$

$$\left(M \xrightarrow{g} \mathrm{Hom}_{Ab}(A,I)\right) \longmapsto \begin{pmatrix} M \longrightarrow I \\ m' \mapsto g(m)(1) \end{pmatrix}$$

$$\begin{matrix} I \\ \downarrow \\ g(m): A \to I \end{matrix}$$

<u>Definition 1.58</u> $F: \mathrm{Mod}_A \to \mathrm{Mod}_B$ left exact functor.

<u>$M \in A$-mod. We define $R^iF(M)$ as follows:</u>

( Note that $\tilde{I} = \mathrm{Hom}_{Ab}(A, \mathbb{Q}/\mathbb{Z})$

Choose an injective $A$-module $I^0$ with an injective homomorphism  A-mo

$$M \hookrightarrow I^0, \quad M \hookrightarrow I^0 := \prod_{\mathrm{Hom}_A(M, \tilde{I})} \tilde{I}$$

$$m \longmapsto \prod_{f \in \mathrm{Hom}_A(M, \tilde{I})} f(m)$$

$$0 \to M \to I^0 \to I^0/M \to 0$$

find injective $A$-module $I^1$ with an injective homo $I^0/M \hookrightarrow I^1$.

Then $0 \to M \to I^0 \to I^1$ exact.

$\rightsquigarrow$ can get long exact sequence $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$

with $I^0, I^1, I^2, \ldots$ injective $A$-modules.

$$M \underset{\text{qis}}{\simeq} I^\bullet = (I^0 \to I^1 \to I^2 \to \cdots)$$

Then define $R^iFM = H^i(FI^\bullet) = H^i(\cdots \to FI^{i-1} \xrightarrow{d^i} FI^i \xrightarrow{d^{i+1}} \cdots)$

$$= \frac{\ker d^{i+1}}{\operatorname{Im} d^i}$$

以后证：$R^iFM$ 的定义与 $I^\bullet$ 选取无关.

<u>Remark</u> If $M$ is an injective $A$-module, we can choose injective resolution

$$\begin{tikzcd} I^0 \arrow[r] \arrow[d, equal] & I^1 \arrow[r] \arrow[d, equal] & I^2 \arrow[r] \arrow[d, equal] & \cdots \\ M & 0 & 0 & \end{tikzcd}$$

then $R^0FM = FM$ and $R^iFM = 0$ for all $i \geq 1$.

$$\boxed{\text{以后} \quad S^{-1} \text{ exact and } \otimes\text{: right exact functor}}$$

Snake引理是构造长正合列的工具.

<u>Prop 1.59 (Snake lemma)</u> In $\mathrm{Mod}_A$,

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "u'"] & N \arrow[r, "v'"] & N'' \arrow[r] & 0 \end{tikzcd}$$

with exact rows

then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker f' \arrow[r, "\bar{u}"] & \ker f \arrow[r, "\bar{v}"] & \ker f'' \arrow[dll, rounded corners] \\ & \operatorname{coker} f' \arrow[r, "\bar{u}'"'] & \operatorname{coker} f \arrow[r, "\bar{v}'"'] & \operatorname{coker} f'' \arrow[r] & 0 \end{tikzcd}$$

where $\bar{u}$ and $\bar{v}$ are restrictions of $u$, 和 $v$

$\bar{u}'$ and $\bar{v}'$ are induced by $u'$ and $v'$.

The boundary homo $\delta$ (connection homo) is defined as follows:

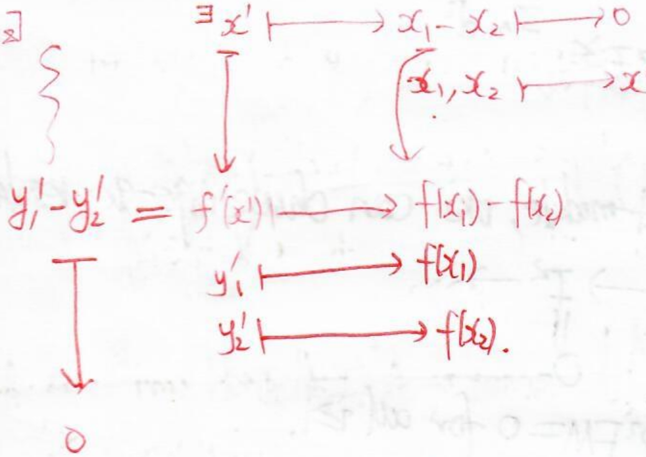
(追图法：diagram chasing)

For $x'' \in \ker f''$,

$$\begin{tikzcd} & & & \exists x \arrow[r, mapsto] \arrow[ddd, mapsto] & x'' & \\ & 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ & 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' & \\ \text{depend on } x \arrow[rr, bend right] & & \exists! y' \arrow[r, mapsto] & f(x) \arrow[r, mapsto] & 0 & \end{tikzcd}$$

Then $\delta(x'') =$ image of $y'$ in $\operatorname{coker}(f')$.

diagram chasing shows that $\delta$ is well-defined, and the long sequence is



利用追图法，可证明如下常用的结论：

**Five lemma 1.60** Consider a diagram of A-modules with exact rows:

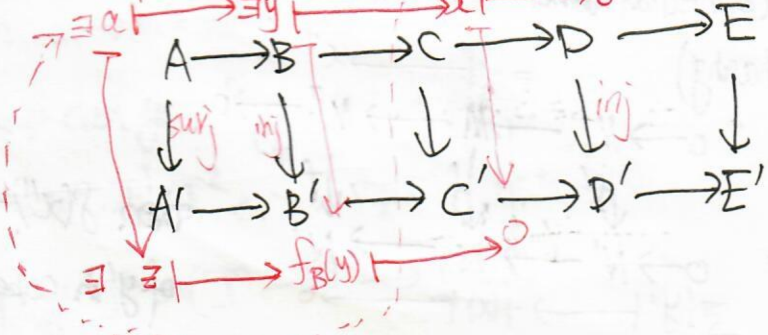
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_A"'] & B \arrow[r] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, "f_C"] & D \arrow[r] \arrow[d, "f_D"] & E \arrow[d, "f_E"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

(1) If $f_A, f_B, f_D, f_E$ are isom, then $f_C$ is an isom.

(2) If $f_B$ and $f_D$ are injective, and if $f_A$ surjective, then $f_C$ is injecti

(3) 若 $f_B$ and $f_D$ are surj, and if $f_E$ is inj, then $f_C$ is surjective.

比如，为证 $f_C$ 单，可如下追图：设 $x \in C$ s.t $f_c(x) = 0$.



# §2 Localization (2024.03.20)

**Why localization?** $X = \operatorname{Spec} A$, ring spectrum.

$\mathfrak{p} = x \in X$ prime ideal of $A$ (point of $X$)

"holomorphic function around $x$" $= A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$

"holomorphic function on $\underset{\substack{\uparrow \\ \text{principal open subset}}}{D(f)}$" $= A_f$ for any $\overset{\neq 0}{f} \in A$.

**Definition 2.1** Say $S \subseteq A$ is a multiplicatively closed subset of $A$ iff $1 \in S$ and $S$ is closed under multiplication.

**Basic example** $f \in A$, $\{f^n\}_{n \geq 0}$ is a multiplicative closed subset.

$\mathfrak{p} \subseteq A$ prime ideal, then $A \setminus \mathfrak{p}$ is multi.

**Prop + Def 2.2** $S \subseteq A$ multiplicatively closed subset. Then there is a ring $S^{-1}A$ together with a ring homo $A \xrightarrow{f} S^{-1}A$ such that

(1) $\forall s \in S$, $f(s)$ is a unit in $S^{-1}A$.

(2) If $g: A \to B$ is a ring hom such that $g(s)$ is a unit for all $s \in S$, then there is a unique ring homo $h: S^{-1}A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ S^{-1}A \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

**Construction of $S^{-1}A$:**

$$S^{-1}A = \frac{A \times S}{\underset{\substack{\uparrow \\ (a_1, s_1) \sim (a_2, s_2) \overset{\text{def}}{\Longleftrightarrow} \exists s \in S \text{ such that } s(s_2 a_1 - s_1 a_2) = 0}}{\sim}} = \left\{ \frac{a}{s} \,\middle|\, \frac{a}{s} \text{ is the equivalence class of } (a, s) \right\}$$

can check: $\sim$ is an equivalence relation on $A \times S$.

Then $S^{-1}A$ is a ring
$$\begin{cases} \dfrac{a_1}{s_1}+\dfrac{a_2}{s_2} = \dfrac{a_1s_2+a_2s_1}{s_1s_2} \\ \dfrac{a_1}{s_1}\cdot\dfrac{a_2}{s_2} = \dfrac{a_1a_2}{s_1s_2} \\ \text{unit } \dfrac{1}{1} \end{cases}$$

with ring homomorphism
$$\begin{aligned} A &\longrightarrow S^{-1}A \\ a &\longmapsto \frac{a}{1} \end{aligned}$$

Note that for $s\in A$, $\frac{s}{1}$ has an inverse $\frac{1}{s}$ in $S^{-1}A$.

We call $S^{-1}A$ the ring of fractions of $A$ w.r.t $S$.

$A\longrightarrow S^{-1}A$ may not injective

For $a\neq 0$, $\frac{a}{1}=0$ in $S^{-1}A \Longleftrightarrow \exists s\in S$ such that $sa=0$
(此时 s 是零因子)

若 s 幂零, 则 $S^{-1}A=0$ (因为 $\frac{a}{t}=0 \overset{\Leftarrow}{} (\exists N, s^N a=0)$)

Universal prop
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ S^{-1}A \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$
$a \mapsto \frac{a}{1}$

If $g(s)$ is a unit for all $s\in S$,
then define $h(\frac{a}{s}) = \frac{g(a)}{g(s)} = g(a)g(s)^{-1}$!

Such $h$ is unique!

Example 2.3 (1) $\mathfrak{p}\in\operatorname{Spec}A$, define $A_\mathfrak{p} := (A\setminus\mathfrak{p})^{-1}A$. (called the localiz
of A at

We show $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

For any $\frac{a}{t}\notin \mathfrak{p}A_\mathfrak{p}$, then $a\notin\mathfrak{p}$. $\Rightarrow a\in A\setminus\mathfrak{p} \Rightarrow \frac{a}{t}$ is a u
$(a\in A, t\in A\setminus\mathfrak{p})$ in $\mathfrak{p}A_\mathfrak{p}$.

The residue field of $A_\mathfrak{p}$ $= A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p}) =$ residue f
of A at

$$= \frac{(A\setminus\mathfrak{p})^{-1}A}{(A\setminus\mathfrak{p})^{-1}\mathfrak{p}}$$

若 A 是整环 $\Rightarrow (0)$ is a prime ideal

$\Rightarrow (A\setminus\{0\})^{-1}A =: \mathrm{Frac}A$ fraction field of $A$.

(2) $S^{-1}A = 0$ iff $0 \in S$ ($\frac{1}{1} = 0$ in $S^{-1}A \Leftrightarrow \exists u \in S$ s.t. $u = u \cdot 1 = 0$)

(3) For $f \in A$, write $A_f = (\{f^n\}_{n \geq 0})^{-1}A = A[\frac{1}{f}] = \frac{A[X]}{(Xf-1)}$

(4) For any ideal $I \subseteq A$, $1+I$ is multiplicatively closed.

(5) For any prime $(p) \in \mathrm{Spec}\,\mathbb{Z}$, $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid (n,p)=1\}$.

Definition 2.4 $S \subseteq A$ multi closed. We have a funtor

$$S^{-1} : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{S^{-1}A}$$

$$M \longmapsto S^{-1}M$$ (defined similarly as $S^{-1}A$)

such that $S^{-1}M$ is an $S^{-1}A$-module.

For $M \xrightarrow{f} N$, by universal property of localization, we get

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[d] \\ S^{-1}M \arrow[r, "S^{-1}f"] & S^{-1}N \end{tikzcd}$$

$S^{-1}A$-module homo

In particular, can define $M_{\mathfrak{p}}$ and $M_f$.

Prop 2.5 (1) $S^{-1} : \mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ exact funtor

In particular, $S^{-1}(M/N) = S^{-1}M/S^{-1}N$ ($0 \to M \to N \to M/N \to 0$)

For $N, P \subseteq M$, since $0 \to N \cap P \to M \to M/N \times M/P$

get $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$, $S^{-1}(N+P) = S^{-1}N + S^{-1}P$

($N+P = \mathrm{Im}(N \oplus P \to M)$)

For $M$ of finite presentation (也即 $\exists\, A^p \longrightarrow A^q \longrightarrow M \longrightarrow 0$)

then $S^{-1}\mathrm{Hom}_A(M,N) \xrightarrow{\cong} \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$

[利用 $S^{-1}$ 正合 以及 $\mathrm{Hom}_A(-,N)$ 左正合].

(pf) For $M' \xrightarrow{f} M \xrightarrow{g} M''$ exact,

we show $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ exact

Since $g \circ f = 0 \Rightarrow S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$

Thus $\mathrm{Im}\, S^{-1}f \subseteq \ker S^{-1}g$

We show $\ker(S^{-1}g) \subseteq \mathrm{Im}\, S^{-1}f$

For $\frac{m}{s} \in S^{-1}M$, If $S^{-1}(g)\left(\frac{m}{s}\right) = \frac{g(m)}{s} = 0$ in $S^{-1}M''$

$\Rightarrow \exists\, t \in S$ s.t $t g(m) = 0 = g(tm)$ in $S^{-1}M''$

$\Rightarrow tm \in \ker g = \mathrm{Im} f$

$tm = f(m')$

$\Rightarrow \frac{m}{s} = \frac{tm}{ts} = \frac{f(m')}{ts} \in S^{-1}M$

$\in \mathrm{Im}(S^{-1}f)$.

(2) If $x \in M$ is zero in $M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$, then $x = 0$.

In other words, $M \longrightarrow \prod_{\mathfrak{m} \in \mathrm{Max}(A)} M_{\mathfrak{m}}$ is injective

In particular, TFAE:

(i) $M = 0$

(ii) $M_{\mathfrak{p}} = 0$ $\forall\, \mathfrak{p} \in \mathrm{Spec} A$

(iii) $M_{\mathfrak{m}} = 0$ $\forall\, \mathfrak{m} \in \mathrm{Max}(A)$.

(pf) If $0 \neq x \in M$ such that $x = 0$ in $A_{\mathfrak{m}}$ ($\forall\, \mathfrak{m} \in \mathrm{Max}(A)$). Let $I = \mathrm{Ann}$

$x \neq 0 \Rightarrow I \neq (1)$. $\Rightarrow \exists$ maximal ideal $\mathfrak{m}$ s.t $I \subseteq \mathfrak{m}$.

Consider $\frac{x}{1} = 0$ in $M_{\mathfrak{m}} \Rightarrow x$ is killed by some element in $A \setminus \mathfrak{m}$ 与 $\mathrm{Ann}$

(3). As a corollary of (2), for any A-module homo $\phi: M \longrightarrow N$,

$\phi$ is injective $\Longleftrightarrow$ $\phi_{\mathfrak{p}}$ injective for all $\mathfrak{p} \in \operatorname{Spec} A$

$\Longleftrightarrow$ $\phi_{\mathfrak{m}}$ injective for all $\mathfrak{m} \in \operatorname{max}(A)$.

Same for surjective/isomorphism.

**Remark 2.6** inj/surj/isom/flatness …. are local properties in the following sense: A property ~~prop~~ about $\left( \begin{array}{l} \text{or an A-module } M \\ \text{a ring } A \end{array} \right.$ is said to be a

Local property iff $\left( \begin{array}{l} A \text{ has } \underline{\text{prop}} \text{ iff } A_{\mathfrak{p}} \text{ has } \underline{\text{prop}} \text{ for all } \mathfrak{p} \in \operatorname{Spec} A \\ M \text{ has } \underline{\text{prop}} \text{ iff } M_{\mathfrak{p}} \text{ ———————} \end{array} \right)$

Now we discuss the prime ideals in $S^{-1}A$.

**Prop 2.7** (1) $\left\{ \begin{array}{c} \text{prime ideals} \\ \text{in } S^{-1}A \end{array} \right\} \xleftrightarrow{1:1} \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \phi \}$

(2) $S^{-1}(\sqrt{I}) = \sqrt{S^{-1}I}$ for any $I \subseteq A$ ideal.

(3). $A \longrightarrow S^{-1}A$, $I \subseteq A$ ideal.

$I^e := I(S^{-1}A)$ extended ideal

$I^e = (1) \Longleftrightarrow I \cap S \neq \phi$.

proof (3). $I^e = (1) \Longleftrightarrow \begin{array}{c} \exists a \in I, s \in S \\ 1 = \frac{a}{s} \text{ in } S^{-1}A \end{array} \Longleftrightarrow \exists t \in S$ s.t $\begin{array}{l} t(a-s) = 0 \\ \Rightarrow ta \in I \cap S \end{array}$ ~~$a = ts$~~ $(\Rightarrow \cancel{a} \in I \cap S)$

(2) by def.

(1). $f: A \longrightarrow S^{-1}A$, ${}^a f: \operatorname{Spec} S^{-1}A \longrightarrow \operatorname{Spec} A$.

眉!

If $\mathfrak{q}$ is a prime ideal in $S^{-1}A$, then ${}^af(\mathfrak{q}) = f^{-1}(\mathfrak{q})$ is a prime ideal in $A$ and $f^{-1}(\mathfrak{q}) \cap S = \phi$.

Conversely, if $\mathfrak{p} \subseteq A$ is a prime ideal with $\mathfrak{p} \cap S = \phi$.

We show $S^{-1}\mathfrak{p}$ is a prime ideal.

This follows from $S^{-1}A / S^{-1}\mathfrak{p} \cong \overline{S}^{-1}(A/\mathfrak{p})$ is integral domain

$\overline{S}$ = image of $S$ in $A/\mathfrak{p}$. ≡

<u>Corollary 2.8</u> If $\mathfrak{p} \in \operatorname{Spec} A$, then

$$\{\text{prime ideals in } A_{\mathfrak{p}}\} \overset{2.7}{\underset{1:1}{=\!=\!=}} \left\{\begin{matrix}\text{prime ideals of } A \\ \text{且与} A \setminus \mathfrak{p} \text{无交}\end{matrix}\right\} \overset{1:1}{=\!=\!=} \left\{\begin{matrix}\text{prime ideals of} \\ A \text{ contained in}\end{matrix}\right.$$

对比 $\operatorname{Spec} A_{\mathfrak{p}}$ 与 $\operatorname{Spec} A/\mathfrak{p}$

<u>Prop 2.9</u> Let $S \subseteq A$ be multi closed and $M$: f.g. $A$ module.

Then $S^{-1}(\operatorname{Ann}(M)) = \operatorname{Ann}(S^{-1}M)$.

Pf Let $M = Ax_1 + \cdots + Ax_n$.

$$\begin{aligned} S^{-1}\operatorname{Ann}(M) &= S^{-1}(\operatorname{Ann}(x_1) \cap \cdots \cap \operatorname{Ann}(x_n)) \\ &= S^{-1}\operatorname{Ann}(x_1) \cap \cdots \cap S^{-1}\operatorname{Ann}(x_n) \\ &= \operatorname{Ann}(S^{-1}x_1) \cap \cdots \cap \operatorname{Ann}(S^{-1}x_n) \\ &= \operatorname{Ann}(S^{-1}x_1 + \cdots + S^{-1}x_n) = \operatorname{Ann}(S^{-1}M) \end{aligned}$$ ≡

Since $(N:P) = \operatorname{Ann}\left(\frac{N+P}{N}\right)$, we get

<u>Corollary 2.10</u> If $N, P \subseteq M$ are submodules, and if $P$ is f.g

Then $S^{-1}(N:P) = (S^{-1}N : S^{-1}P)$.

问题：如果去掉 $f\cdot g$ 这一条件，结论是否成立？

## §3 Tensor product and flatness

Definition 3.1 $M, N, P \in \mathrm{Mod}_A$. A map $f: M\times N \longrightarrow P$ is A-bilinear if $\forall\, x \in M$, the map $\begin{array}{l} N \longrightarrow P \\ y \longmapsto f(x,y) \end{array}$ is A-linear, and for all $y \in N$, the map $\begin{array}{l} M \longrightarrow P \\ x \longmapsto f(x,y) \end{array}$ is A-linear.

M与N的 tensor product $T = M\otimes_A N$ 满足：

$$\left\{ \begin{array}{c} M\times N \longrightarrow P \\ \text{bilinear} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{A-linear map} \\ M\otimes N \to P \end{array} \right\} \text{ for all } P \in \mathrm{Mod}_A.$$

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Prop 3.2 $M, N$: A-module. Then $\exists$ pair $(T, g)$: $T$ is an A-module, $g: M\times N \to T$ A-bilinear, satisfying the following property:

Given any A-module $P$, and any A-linear map $f: M\times N \longrightarrow P$, there is a unique A-linear mapping $f': T \longrightarrow P$ such that $f = f'\circ g$.

$$\begin{tikzcd} M\times N \arrow[r, "f"] \arrow[d, "g"'] & P \\ T \arrow[ur, dashed] & \end{tikzcd}$$

Moreover, if $(T, g)$ and $(T', g')$ are two pairs with this property, then $\exists!$ isom $j: T \longrightarrow T'$ such that $j\circ g = g'$.

We call $(T, g)$ the tensor product of $M$ and $N$, and denote $T$ by $M\otimes_A N$, and write $\begin{array}{l} M\times N \longrightarrow M\otimes_A N \\ (m,n) \longmapsto m\otimes n. \end{array}$

proof (1) 唯一性

$$\begin{tikzcd} M\times N \arrow[r, "g"] \arrow[d, "g'"'] & T \arrow[dl, dashed, bend left, "\exists! j"] \\ T' \arrow[ur, dashed, "\exists! j'"] & \end{tikzcd}$$

such that $g'=j\circ g$ and $g=j'\circ g'$.

$\Rightarrow g'=j\circ j'\circ g'$

$$\begin{tikzcd} M\times N \arrow[r, "g'"] \arrow[d, "g'"'] & T' \\ T' \arrow[ur, shift left, "id"] \arrow[ur, shift right, "j\circ j'"'] & \end{tikzcd}$$

泛性质唯一性 $\Rightarrow j\circ j'=id$

同理 $j'\circ j=id$ $\Big\} \Rightarrow j$

(2) 存在性

$$C=A^{(M\times N)} = \text{由}M\times N\text{中元素生成的自由}A\text{-模}$$
$$=\left\{\sum_{i=1}^{n} a_i(x_i,y_i) \;\middle|\; \begin{matrix} x_i\in M \\ y_i\in N \\ a_i\in A \end{matrix} \quad n\in\mathbb{N}\right\}$$

$D\subseteq C$ submodule generated by elements of $C$ of the following type

$$(x+x',y)-(x,y)-(x',y)$$
$$(x,y+y')-(x,y)-(x,y')$$
$$(ax,y)-a(x,y)$$
$$(x,ay)-a(x,y).$$

Now let $T=C/D$. For $(x,y)\in C$, denote $x\otimes y=$ image of $($

then $T$ is generated by $\{x\otimes y \mid x\in M, y\in N\}$ with relations

$$(x+x')\otimes y = x\otimes y + x'\otimes y$$
$$x\otimes(y+y') = x\otimes y + x\otimes y'$$
$$(ax)\otimes y = x\otimes ay = a(x\otimes y)$$

The map $\begin{matrix} M\times N \xrightarrow{g} T \\ (x,y)\longmapsto x\otimes y \end{matrix}$ is $A$-bilinear.

(3) universal property

Any map $f: M\times N \longrightarrow P$ of A-modules extends by linearity to A-module homo $\bar{f}: C \longrightarrow P$.

If $f$ is A-bilinear, then $\bar{f}$ vanishes on $D$

$\Rightarrow \bar{f}$ induce a A-homo $f': C/D \longrightarrow P$ s.t $f'(x\otimes y) = f(x,y)$.

The map $f'$ is uniquely defined by this condition, and therefore the pair $(T,g)$ satisfies the universal property. $\square$

Remark 3.3 (1) By construction, $M\otimes N$ is generated by $\{x\otimes y \mid x\in M, y\in N\}$.

If M and N are finitely generated, the $M\otimes N$ is also f.g.

(2) $\otimes$ does not preserve injective map / not exact / but right exact.

$2\mathbb{Z} \hookrightarrow \mathbb{Z}$ injective $\left(\mathbb{Z} \xrightarrow[\text{injective}]{\times 2} \mathbb{Z}\right)$,

but

$$\begin{tikzcd}
\mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[d, equal] \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

is the zero map.

(3) Universal property is more important than the "具体构造".

Corollary 3.4 Let $x_i \in M$ and $y_i \in N$ s.t $\sum x_i\otimes y_i = 0$ in $M\otimes N$.
Then $\exists$ f.g. submodule $M_0 \subseteq M$ and $N_0 \subseteq N$ s.t $\sum x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.

proof If $\sum x_i \otimes y_i = 0$, then $\sum (x_i, y_i) \in D$.

$\Rightarrow \sum(x_i, y_i)$ is a finite sum of generators of $D$.

Let $M_0 \subseteq M$ be the submodule generated by the $x_i$ and all the elements of $M$ which occurs as first coordinates in those generators of $D$, and define $N_0 \subseteq N$ similarly. Then $\sum x_i \otimes y_i = 0$ as an element of $M_0 \otimes N_0$ $\blacksquare$

类似可定义 multilinear map $f: M_1 \times \cdots \times M_r \longrightarrow P$ 以及 multitensor product

类似可证明:

<u>Prop 3.5</u> $\exists!$ pair $(T, g)$ $\left\{\begin{array}{l} T: A\text{-module} \\ g: M_1 \times \cdots \times M_r \longrightarrow T \text{ multilinear such that} \end{array}\right.$

$$\begin{tikzcd}
\forall\, M_1 \times \cdots \times M_r \arrow[r, "\text{multilinear}"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & P \\
T \arrow[ur, dashed, "\exists!\, f\ A\text{-linear}"'] &
\end{tikzcd}$$

<u>Exercise 3.6</u> $M, N, P \in \mathrm{Mod}_A$. Then

(1) $M \otimes_A N \xrightarrow{\sim} N \otimes_A M$

$(x \otimes y) \longmapsto y \otimes x$

(2) $(M \otimes N) \otimes P \xrightarrow{\sim} M \otimes (N \otimes P) \xrightarrow{\sim} M \otimes N \otimes P.$

(3) $(M \oplus N) \otimes P \xrightarrow{\sim} (M \otimes P) \oplus (N \otimes P)$

(4) $A \otimes_A M \xrightarrow{\cong} M$

$a \otimes x \longmapsto ax.$

$\quad \|$

$1 \otimes ax.$

<u>Exercise 3.7</u> $A, B$: rings. Then $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$

where $M \in$ ~~A~~ $\mathrm{Mod}_A$, $P \in \mathrm{Mod}_B$, $N$: $(A, B)$-bimodule s.t.

$a(xb) = (ax)b$, $\forall\, a \in A$, $x \in$

$b \in$

<u>Construction 3.8</u> $M \xrightarrow{f} M'$, $g: N \longrightarrow N'$ hom of A-modules.

define $M \times N \xrightarrow{h} M' \times N'$ by $h(x, y) = f(x) \otimes g(y)$.

$h$ is bilinear, induces $f \otimes g: M \otimes N \longrightarrow M' \otimes N'$ s.t. $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$

For $\begin{array}{l} M \xrightarrow{f} M' \xrightarrow{f''} M'' \\ N \xrightarrow{g} N' \xrightarrow{g'} N'' \end{array}$, then $(f' \circ f) \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ agree.

all elements of the form $x \otimes y \in M \otimes N$.

Since $\{x \otimes y\}$ generates $M \otimes N$, we have $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.

## 3.9 Restriction of Scalars

$f: A \to B$ homo of rings, $N$: $B$-module.

Then $N$ has an $A$-module structure via $A \times N \longrightarrow B \times N \longrightarrow N$.
$a.n = f(a).n$

This $A$-module is said to be obtained from $N$ by restriction of scalars:

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_A.$$

Prop 3.10 $A \to B$ ring homo. Suppose $N$: f.g. as $B$-module and that $B$ is f.g as $A$-module. Then ~~B~~ $N$ is f.g. as an $A$-module.

proof $\exists\, B^{r_1} \twoheadrightarrow N$, $\exists\, A^{r_2} \twoheadrightarrow B$ $\Rightarrow$ $A^{r_1 r_2} \twoheadrightarrow N$. □

## 3.11 Extension of Scalars

$M \in \mathrm{Mod}_A$, $A \to B$ ring homo. Can form $A$-module $M_B = B \otimes_A M$, which is also a $B$-module by $b(b' \otimes x) = bb' \otimes x$.

Call $M_B$ the $B$-module obtained from $M$ by extension of scalars.

Prop 3.12 If $M$ is f.g. $A$-module, then $M_B$ is a f.g. $B$-module.

We consider the special case: $A \longrightarrow S^{-1}A$.

Prop 3.13 $M \in \mathrm{Mod}_A$. $S \subseteq A$ multiplicative set.

Then $S^{-1}A \otimes_A M \xrightarrow{\overset{f}{\simeq}} S^{-1}M$ as $S^{-1}A$-module

$$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$$

$a \in A, s \in S, m \in M$

(Note that $f$ is induced by the $A$-bilinear map $S^{-1}A \times M \longrightarrow S^{-1}M$,
$(\frac{a}{s}, m) \longmapsto \frac{am}{s}$

proof 先证明 $S^{-1}A\otimes_A M$中元素都具有形式 $\frac{1}{s}\otimes m$.

Let $\sum_i \frac{a_i}{s_i}\otimes m_i \in S^{-1}A\otimes_A M$.

Put $s=\prod s_i\in S$, $t_i=\prod_{j\neq i} s_j$.

Then $\sum_i \frac{a_i}{s_i}\otimes m_i = \sum_i \frac{a_i t_i}{s}\otimes m_i = \sum_i \frac{1}{s}\otimes a_i t_i m_i$

$= \frac{1}{s}\otimes \sum a_i t_i m_i$.

<u>f is surjective</u> (clear)

<u>f is injective</u> Suppose that $f(\frac{1}{s}\otimes m)=0 \Rightarrow \frac{m}{s}=0 \Rightarrow \exists\, t\in S$ s.t $tm=$

$\Rightarrow \frac{1}{s}\otimes m = \frac{t}{ts}\otimes m = \frac{1}{ts}\otimes tm = 0 \Rightarrow f$ inj.

By Prop 3.13, we have

<u>Prop 3.14</u> $M, N$: A-Mod. There is a unique isomorphism of $S^{-1}A$-module

$$f: S^{-1}M\otimes_{S^{-1}A} S^{-1}N \xrightarrow{\sim} S^{-1}(M\otimes_A N)$$

$$\frac{m}{s}\otimes\frac{n}{t} \longmapsto \frac{m\otimes n}{st}$$

In particular, if $\mathfrak{p}$ is a prime ideal, then $M_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong (M\otimes_A N)_{\mathfrak{p}}$

## 3.15 Hom and ⊗ are adjoint pairs

$M, N, P\in \mathrm{Mod}_A$. We construct a canonical isomorphism

$$\begin{tikzcd}
\mathrm{Hom}(M\otimes N, P) \arrow[rr, "\cong"] \arrow[dr, "\cong"', "\text{by def of } \otimes"] & & \mathrm{Hom}(M, \mathrm{Hom}(N,P)) \arrow[dl, "\cong", "\text{clear.}"'] \\
& \left\{\begin{matrix}\text{A-bilinear maps} \\ M\times N\longrightarrow P\end{matrix}\right\} &
\end{tikzcd}$$

as follows: (以下可不讲)

— Given $M\otimes N \xrightarrow{f} P$ 记 $M\times N\xrightarrow{f} P$ A-bilinear

$\rightsquigarrow$ A-linear $M \longrightarrow \mathrm{Hom}(N,P)$

$m \longmapsto (f(m,-): N\to P)$

Conversely, given $\phi: M \longrightarrow \mathrm{Hom}_A(N,P)$ A-linear.

define a A-linear map $M\times N \xrightarrow{f} P$ by $(m,n) \longmapsto \phi(m)(n)$.

$T =$ "$-\otimes N$" 是左伴随 (右正合函子)

$U =$ "$\mathrm{Hom}(N,-)$" 是右伴随 (左正合函子)

$$\mathrm{Hom}(TM, P) = \mathrm{Hom}(M, U(P)).$$

Prop 3.16 Let $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ exact in $\mathrm{Mod}_A$.

$N \in \mathrm{Mod}_A$. Then the sequence

$$(*) \qquad M'\otimes N \xrightarrow{f\otimes 1} M\otimes N \xrightarrow{g\otimes 1} M''\otimes N \longrightarrow 0 \quad \text{is exact}$$

(i.e., $-\otimes N: \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A$ is right exact).

proof 记 $E = (M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0)$.

记 $(*)$ 为 $E\otimes N$.

since $\mathrm{Hom}(-,\mathrm{Hom}(N,P))$ left exact

P: any A-module. Then E exact $\Longrightarrow \mathrm{Hom}(E, \mathrm{Hom}(N,P))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}(E\otimes N, P)$ is exact

$\Rightarrow E\otimes N$ is exact by "exact test" prop 1.47. ▦

Remark 3.17 $-\otimes N$ is not an exact functor in general.

$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$ exact, but $0 \longrightarrow \mathbb{Z}\otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z}$ is not exact

$x \longmapsto 2x \qquad\qquad x\otimes y \longmapsto 2x\otimes y = x\otimes 2y = 0.$

Prop + Def 3.18 TFCAE for $N\in \mathrm{Mod}_A$:

(1) N is flat, i.e., $-\otimes N$ is exact (定义)

(2) If $0\to M' \to M \to M'' \to 0$ exact, then $0 \to M'\otimes N \to M\otimes N \to M''\otimes N \to 0$ exact

(3) If $f: M' \longrightarrow M$ injective, then $f\otimes 1: M'\otimes N \to M\otimes N$ injective.

(4) If $f: M' \to M$ injective with $M'$ and $M$ are f.g, then $f\otimes 1: M'\otimes N \to M\otimes N$ is injective.

proof (1) ⇔ (2) by def.

(2) ⇔ (3) by Prop 3.16

(3) ⇒ (4) clear

(4) ⇒ (3) If $f: M' \to M$ injective and $u = \sum x_i' \otimes y_i \in \mathrm{Ker}(f\otimes 1)$, that $\sum f(x_i') \otimes y_i = 0$ in $M\otimes N$.

Let $M_0' \subseteq M'$ be the submodule generated by $x_i'$

$$u_0 = \sum x_i' \otimes y_i \text{ in } M_0' \otimes N.$$

By Prop 3.4, $\exists$ f.g. submodules $M_0 \subseteq M$ containg $f(M_0')$ and that

$$\sum f(x_i') \otimes y_i = 0 \text{ in } M_0 \otimes N.$$

If $f_0: M_0' \to M_0$ is the restriction of $f$, then $(f_0 \otimes 1)(u_0) = 0$

Since $M_0$ and $M_0'$ are f.g, $f_0 \otimes 1$ is injective and therefore $u_0 = 0 \Rightarrow$

Flatness is a local property.

Prop 3.19 For any A-module $M$. TFAE:

(1) $M$ is a flat A-module.

(2) $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Spec}A$.

(3) $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}(A)$.

proof (1) ⇒ (2) by exactness of the localization functor.

(2) ⇒ (3) clear.

(3) ⇒ (1). If $N \to P$ homo of A-modules,

$$N \to P \text{ inj} \Leftrightarrow N_\mathfrak{m} \to P_\mathfrak{m} \text{ inj for all maximal } \mathfrak{m}$$

$$\begin{array}{ccc} \Rightarrow N_\mathfrak{m} \otimes_{A_\mathfrak{m}} M_\mathfrak{m} & \longrightarrow & P_\mathfrak{m} \otimes_{A_\mathfrak{m}} M_\mathfrak{m} \text{ inj by flatness} \\ \| & & \| \\ (N \otimes_A M)_\mathfrak{m} & & (P \otimes_A M)_\mathfrak{m} \end{array} \Rightarrow N \otimes_A M \to P \otimes_A M \text{ injective}$$

Exercise 3.20 $A \xrightarrow{f} B$ ring homo. M: flat A-module. Then $M_B = B\otimes_A M$ is a flat B-module.

$\big[ 0 \to N \to P$ inj in $\mathrm{Mod}_B$, $N\otimes_B M_B = N\otimes_B (B\otimes_A M) = \cancel{N\otimes}\, N\otimes_A M \big)$

Remark 3.21 For $0 \to N \to P \to Q \to 0$ exact, M: A-module,

we have a long exact seq

$$\begin{tikzcd}
 N\otimes M \arrow[r] & P\otimes M \arrow[r] & Q\otimes M \arrow[r] & 0 \\
 \mathrm{Tor}_1^A(N,M) \arrow[r] & \mathrm{Tor}_1^A(P,M) \arrow[r] & \mathrm{Tor}_1^A(Q,M) \arrow[ull] & \\
 & & \mathrm{Tor}_2^A(Q,M) \arrow[ull] &
\end{tikzcd}$$

(obstruction of flatness)

$\mathrm{Tor}_n^A(-,M)$ n-th left derived funtor of $-\otimes_A M$.

For M flat (or injective), we have $\mathrm{Tor}_i^A(-,M)=0$ for $i>0$.

## 3.22 Algebra

2024.03.27

$A \xrightarrow{f} B$ ~~righ~~ ring homomorphism

View B as an A-module by $\begin{array}{l} A\times B \longrightarrow B \\ (a,b) \longmapsto f(a)b \end{array}$

The ring B, equipped with this A-module structure, is said to be an A-algebra.

By def: A-algebra = ring B with a ring homo $A \xrightarrow{f} B$.

— Every ring A is a $\mathbb{Z}$-algebra by $\begin{array}{l} \mathbb{Z} \longrightarrow A \\ n \longmapsto n\cdot 1 \end{array}$

— If $A=K$ is a field, $B\neq 0$, then f is injective.

$\Rightarrow$ K-algebra = a ring containing K as a subring.

B, C: A-algebras.

an A-algebra homo $B \xrightarrow{h} C$ is a ring homomorphism, which is also an A-module homo.

$$\begin{tikzcd}
B \arrow[rr, "h"] & & C \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

- We say a ring $f: A \to B$ is finite (or $B$ is a finite $A$-algebra) if $B$ finitely generated as an $A$-module.
- We say a ring homo $f: A \to B$ is of finite type (and $B$ is a f.g $A$-algebra) if $\exists$ finite set $\{x_1, \cdots, x_n\} \subseteq B$ such that

$$\begin{aligned} A[T_1, \cdots, T_n] &\twoheadrightarrow B \text{ surjective} \\ T_i &\longmapsto x_i \end{aligned}$$

Ring $A$ is said to be f.g if it is a f.g as a $\mathbb{Z}$-algebra.

Tensor product of algebras

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$ ring homo $\Rightarrow D = B \otimes_A C$ is an $A$-module.

multiplication on $D = B \otimes_A C$

$$\begin{aligned} (B \times C) \times (B \times C) &\longrightarrow D \quad \text{multilinear} \\ (b, c) \times (b', c') &\longmapsto bb' \otimes cc' \end{aligned}$$

$$\begin{tikzcd} B \otimes C \otimes B \otimes C \arrow[r] \arrow[d, phantom, "\wr"] & D \\ D \otimes D \arrow[ru] & \end{tikzcd}$$ $A$-module homo

$\Leftrightarrow \mu: D \times D \longrightarrow D$ $A$-bilinear

$D$ is a comm. ring with unit $1 \otimes 1$

moreover $D$ is an $A$-algebra $A \longrightarrow D$

$$a \longmapsto f(a) \otimes 1 = 1 \otimes g(a).$$

## Flatness (平坦)

**Def 3.23** $M \in \mathrm{Mod}_A$

(1) $M$ is a flat $A$-module if $- \otimes_A M : \mathrm{Mod}_B \to \overset{Ab}{\mathrm{Mod}_B}$ is exact

(2) $M$ is a faithful flat $A$-module if (an seq $E$ is exact iff $E \otimes M$ is exact)

**Example 3.24** (1) Free modules are faithful flat.

(2) Projective modules (自由模, 自如环) are flat (but not the converse, e.g. $\mathbb{Q}$ as $\mathbb{Z}$-module).

(3) $A = B \times C$, $B, C$ are ring. Then $B$ is a projective $A$-module, hence flat over $A$, but $B$ is not f.f over $A$.

**Theorem 3.25** TFCAE:

(0) $M$ is $A$-flat.

(1) For any $A$-module $N$, we have $\mathrm{Tor}_1^A(M,N)=0$

(2) If $0 \to N' \to N$ is an exact seq of $A$-modules, then $0 \to N' \otimes M \to N \otimes M$ is exact.

(3) For any f.g ideal $I \subseteq A$, the seq $0 \to I \otimes M \to M$ is exact, i.e., $I \otimes M \simeq IM$.

(4) $\mathrm{Tor}_1^A(M, A/I) = 0$ for any f.g ideal $I$ of $A$.

(5) $\mathrm{Tor}_1^A(M,N)=0$ for any finite $A$-module $N$.

(6) If $a_i \in A$, $x_i \in M$ $(1 \le i \le r)$, and $a_1x_1 + \cdots + a_rx_r = 0$, then $\exists s \geq 1$, $b_{ij} \in A$ and $y_j \in M$ $(1 \le j \le s)$ s.t $x_i = b_{i1}y_1 + b_{i2}y_2 + \cdots + b_{is}y_s$ $\forall i$, $a_1b_{1j} + \cdots + a_rb_{rj} = 0$ $\forall j$.

*proof* (0)$\Leftrightarrow$(2) by right exactness of tensor ~~funtor~~ $-\otimes M$.

~~(2)$\Leftrightarrow$(3)~~

(2)$\Rightarrow$(3) Clear.

(3)$\Leftrightarrow$(4) Consider $0 \to I \to A \to A/I \to 0$ exact as $A$-modules.

$$0 \to \mathrm{Tor}_1^A(A/I, M) \to I \otimes M \to M \to A/I \otimes M \to 0$$

$$\Rightarrow 0 = \mathrm{Tor}_1^A(M,A) \longrightarrow \mathrm{Tor}_1^A(A/I, M) \longrightarrow I \otimes M \longrightarrow M \to A/I \otimes M \to 0$$

Thus $0 \to I \otimes M \to M \to \underset{\substack{\wr \\ A/IM}}{A/I \otimes M} \to 0$ exact $\Leftrightarrow \mathrm{Tor}_1^A(A/I, M) = 0$.

$$\begin{tikzcd}(3) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (0) \arrow[d, Leftrightarrow, "去证"] \\ & (2)\end{tikzcd}$$

首先, every ideal of $A$ is the direct limit of f.g. ideals contained in

Since direct limit is exact and $(\varinjlim M_\lambda) \otimes_A N \overset{引题}{\cong} \varinjlim (M_\lambda \otimes_A N)$.

$\Rightarrow$ For any ideal $I \subseteq A$, $I \otimes M \longrightarrow M$ is injective.

moreover, if $N \in \mathrm{Mod}_A$, $N' \subseteq N$ submodule, then since $N$ is the direct limit of modules of the form $N'+F$ with $F$ f.g., to prove that $N' \otimes M \to N \otimes M$ is we may assume that $N = N' + A\omega_1 + \cdots + A\omega_n$.

Put $N_i = N' + A\omega_1 + \cdots + A\omega_i$ $(1 \leq i \leq n)$. We only need to show that each step in the chain $N' \otimes M \longrightarrow N_1 \otimes M \to N_2 \otimes M \to \cdots \to N \otimes M$ is injective,

and finally, we only need to show that:

—— If $N = N' + A\omega$, then $N' \otimes M \longrightarrow N \otimes M$ is injective.

Now we set $I = \{a \in A \mid a\omega \in N'\}$. Then we get the exact seq

$$0 \to N' \to N \to A/I \to 0$$

This induces a long exact sequence

$$\cdots \to \mathrm{Tor}_1^A(M, A/I) \to N' \otimes M \longrightarrow N \otimes M \longrightarrow A/I \otimes M \longrightarrow 0$$

hence it is enough to prove that $\mathrm{Tor}_1^A(M, A/I) = 0$.

Now the result follows from $(3) \Leftrightarrow (4)$.

目前去证 $(0) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

$(0) \Rightarrow (1) \Rightarrow (5)$ Let $\cdots \to L_i \to L_{i-1} \to \cdots \to L_0 \to N \to 0$ be a projective reso of $N$. Then $\cdots \to L_i \otimes M \to \cdots \longrightarrow L_0 \otimes M \to N \otimes M \to 0$ exact

proj

Thus $\mathrm{Tor}_i^A(M, N) = 0$ $\forall i > 0$.

$(5) \Rightarrow (4)$ clear.

$(5) \Rightarrow (1)$ by the proof of $(3) \Rightarrow (0)$.

$(0) \Rightarrow (6)$ Suppose $a_1 x_1 + \cdots + a_r x_r = 0$.

Consider exact sequence $K \overset{g}{\hookrightarrow} A^r \xrightarrow{f} A$

$$(b_1, \ldots, b_r) \longmapsto a_1 b_1 + \cdots + a_r b_r$$

$k = \ker(g)$, $g$: inclusion.

Then $K\otimes M \xhookrightarrow{g} M^r \xrightarrow{f_M} M$ is exact

$(t_1,\dots,t_r) \longmapsto a_1t_1+\dots+a_rt_r$

but $(x_1,\dots,x_r) \in \ker f_M = \mathrm{Im}(K\otimes M \longrightarrow M^r)$

$\Rightarrow (x_1,\dots,x_r) = g\left(\sum_{j=1}^{s} \beta_j \otimes y_j\right)$, $\beta_j \in K$, $y_j \in M$

write $\beta_j = (b_{1j},\dots,b_{rj}) \Rightarrow$ 结论!

(6) ⇒ (3) Let $a_1,\dots,a_r \in I$, and $x_1,\dots,x_r \in M$ s.t. $\sum a_i x_i = 0$ (show $\sum a_i \otimes x_i = 0$)

By assumption, $x_i = \sum b_{ij} y_j$, $\sum a_i b_{ij} = 0$

thus in $I\otimes M$ we have

$$\sum_i a_i \otimes x_i = \sum_i a_i \otimes \sum_j b_{ij} y_j = \sum_j \left(\sum_i a_i b_{ij}\right) \otimes y_j = 0. \quad ∎$$

Prop 3.26 (1) Transitivity $A \xrightarrow{\phi} B$ flat homo of rings (B is a flat A-module)

then a flat B-module N is also flat over A.

pf E: seq of A modules. Then $E\otimes_A N = (E\otimes_A B)\otimes_B N$.

If E exact, then $E\otimes_A B$ exact $\Rightarrow (E\otimes_A B)\otimes_B N = E\otimes_A N$ exact. ∎

(2) Change of base $\phi: A \to B$ ring homo. M: flat A-module.

Then $M\otimes_A B$ is a flat B-module.

pf E: exact seq in $Mod_B$. Then ~~E~~ $E\otimes_B (M\otimes_A B) \cong E\otimes_A M$ exact. ∎

(3) Localization If $S \subseteq A$ multi closed, then $S^{-1}A$ is flat over A.

Prop 3.27 $\phi: A \to B$ flat, $M, N \in Mod_A$. Then $\mathrm{Tor}_i^A(M,N)\otimes_A B \cong \mathrm{Tor}_i^B(M\otimes_A B, N\otimes_A B)$.

(in particular, since $A \to A_{\mathfrak{p}}$ flat $\Rightarrow \mathrm{Tor}_i^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \mathrm{Tor}_i^A(M,N)_{\mathfrak{p}}$ $\forall \mathfrak{p} \in \mathrm{Spec} A$)

proof Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of the A-mod

Since B is A-flat $\Rightarrow \cdots \to P_1 \otimes_A B \to P_0 \otimes_A B \to M \otimes_A B \to 0$ exact

and is a projective resolution of $M \otimes_A B$

( $P_i \otimes_A B$ 是自由B-模直和項 $\Rightarrow P_i \otimes_A B$ proj as B-

thus $\mathrm{Tor}_i^A(M,N) = H_i(P_\bullet \otimes_A N)$

$$\mathrm{Tor}_i^B(M\otimes_A B, N \otimes_A B) = H_i(P_\bullet \otimes_A N \otimes_A B)$$

the exact functor $-\otimes_A B$ commutes with homology $= \frac{\mathrm{Ker}}{\mathrm{Image}}$

$$= H_i(P_\bullet \otimes_A N) \otimes_A B$$

$$= \mathrm{Tor}_i^A(M,N) \otimes_A B. \qquad ∎$$

Exercise 3.28 Assume A is Noetherian.

$\phi: A \to B$ flat, $M, N$: A-module $\boxed{\text{s.t } M \text{ is f.g. over } A}$

Then $\mathrm{Ext}_A^i(M,N) \otimes_A B = \mathrm{Ext}_B^i(M \otimes_A B, N \otimes_A B)$.

In particular, for any finite A-module $M$ over the Noetherian ring A,

$$\mathrm{Ext}^i_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{Ext}_A^i(M,N)_\mathfrak{p}.$$

Prop 3.29. $(A, \mathfrak{m}, k)$ local ring. $M$: A-module. f.g.

Suppose either $\mathfrak{m}$ is nilpotent or $M$ is finite over A.

Then $M$ is free $\Leftrightarrow$ $M$ is projective $\Leftrightarrow$ $M$ is flat.

pf We only need to prove that if $M$ is flat, then it is free.

Let $x_1, \ldots, x_n \in M$ s.t their images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{m}M = M \otimes_A k$ are linearly independent over k, then $\underbrace{x_1, \ldots, x_n}_{\text{we show}}$ are linearly indep over

($\Rightarrow M$ is A-free).

We prove by induction on n. When $n=1$, let $ax_1=0$. Then there exist $y_1,\dots,y_r\in M$, $b_1,\dots,b_r\in A$ such that $ab_i=0$ and such that $x_1=\sum b_i y_i$ (by Prp 3.25)

Since $\bar{x}\neq 0$ in $M/\mathfrak{m}M$, not all $b_i$ are in $\mathfrak{m}$. Suppose $b_1\notin\mathfrak{m}$.

Then $b_1$ is a unit in the local ring A. Since $ab_1=0 \Rightarrow a=0$.

Suppose $n>1$ and $\sum_{i=1}^{n} a_i x_i=0$. Then by 3.25, there exist $y_1,\dots,y_r\in M$ and $b_{ij}\in A\ (1\leq j\leq r)$ such that $x_i=\sum_j b_{ij}y_j$ and $\sum_i a_i b_{ij}=0$

Suppose $x_n\notin\mathfrak{m}M$, we have $b_{nj}\notin\mathfrak{m}$ for at least one j.

Since $a_1b_{1j}+\dots+a_nb_{nj}=0$ and $b_{nj}$ is a unit,

$$\Rightarrow a_n=\sum_{1}^{n-1}c_i a_i \quad (c_i=-b_{ij}/b_{nj})$$

$$\Rightarrow 0=\sum_{1}^{n}a_i x_i=a_1(x_1+c_1x_n)+\dots+a_{n-1}(x_{n-1}+c_{n-1}x_n)$$

Since the elements $\bar{x}_1+\bar{c}_1\bar{x}_n,\dots,\bar{x}_{n-1}+\overline{c_{n-1}}\,\bar{x}_n$ are linearly independent over k, by the induction hypothesis, we get $a_1,\dots,a_{n-1}=0$, and $a_n=\sum_{1}^{n-1}c_i a_i=0$. ▦

Remark 3.30 If M is flat but not finite, then M is not necessarily free (e.g. $A=\mathbb{Z}_{(p)}$, $M=\mathbb{Q}$).

I. Kaplansky: any projective module over a local ring is free.

现处理 faithful flatness. 2024.03.30

Theorem 3.31 A: ring, $M\in\mathrm{Mod}_A$. The following conditions are equivalent:

(1) M is faithfully flat over A, i.e., a complex E is exact iff $E\otimes_A M$ is exact

(2) M is flat over A, and for any A-module $N\neq 0$, we have $N\otimes M\neq 0$.

(3) M is flat over A, and for any maximal ideal $\mathfrak{m}\subseteq A$, we have $\mathfrak{m}M\neq M$.

proof (1) $\Rightarrow$ (2). Suppose $N \otimes M = 0$. Consider $0 \to N \to 0$

As $0 \to N \otimes M \to 0$ exact $\Rightarrow 0 \to N \to 0$ exact $\Rightarrow N = 0$.

(2) $\Rightarrow$ (3). Since $A/m \neq 0 \Rightarrow A/m \otimes M = M/mM \neq 0$

(3) $\Rightarrow$ (2) Take $x \in N$, $x \neq 0$ ~~we show $Ax \otimes M$~~ $\neq 0$ (~~then $N \otimes M \neq 0$~~).

Since $0 \to A/I \otimes M \to N \otimes M$ exact $\Rightarrow$ 只要证明 $A/I \otimes M \neq 0$.

As an A-module, $Ax \cong A/I$ for $I = \mathrm{Ann}(x) \subsetneq A$ ideal of A

Let $m$ be a maximal ideal of A containing $I$,

$\Rightarrow IM \subseteq mM \subseteq M \Rightarrow A/I \otimes M = M/IM \neq 0$.

(2) $\Rightarrow$ (1) $S: N' \xrightarrow{f} N \xrightarrow{g} N''$ seq of A-modules.

Suppose that $S \otimes M : N' \otimes M \xrightarrow{f_M} N \otimes M \xrightarrow{g_M} N'' \otimes M$ exact.

As M is flat, the exact functor transfers kernel into kernel and image into image.

thus $\mathrm{Im}(g \circ f) \otimes M = \mathrm{Im}(g_M \circ f_M) = 0 \underset{\text{by assump}}{\Longrightarrow} \mathrm{Im}(g \circ f) = 0$.

Hence S is a complex, and if $H(S)$ denote its homology at $N$, we have $H(S) \otimes M = H(S \otimes M) = 0$.

By assumption $\Rightarrow H(S) = 0 \Rightarrow S$ exact.

**Corollary 3.32** $A, B$: local rings. $\psi: A \to B$ local homomorphism.

$M \neq 0$ finite B-module.

Then $M$ is flat over $A \Leftrightarrow M$ is f.f over A. (特别可取 $\psi = id$).

In particular, B is flat over A iff it is f.f. over A

proof. $m_A, m_B$: maximal ideals of A, B respectively.

Since $\psi$ local $\Rightarrow m_A M = f(m_A) M \subseteq m_B M$

By Nakayama, $m_B M \neq M$. Thus $m_A M \neq M$. ☐

**Exercise 3.33** (1) Faithful-flatness is transitive ($B$ is f.f. $A$-algebra, $M$ is f.f $B$-module then $M$ is f.f over $A$)

(2) —— is preserved by base change: $M$ is f.f over $A$, and $B$ is any $A$-algebra, then $M\otimes_A B$ is f.f $B$-module.

(3) —— has the following descent property: If $B$ is an $A$-algebra and if $M$ is a f.f $B$-module which is also f.f over $A$, then $B$ is f.f over $A$.

**Prop 3.34** $\varphi : A \to B$ f.f homomorphism of rings. Then

(1) For any $A$-module $N$, the map $N \to N\otimes B$ defined by $x \mapsto x\otimes 1$ is injective. In particular, $\varphi$ is injective and $A$ can be viewed as a subring of $B$

(2) For any ideal $I \subseteq A$, we have $IB \cap A = I$.

(3) ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

**proof** (1) Let $0 \neq x \in N$. Then $0 \neq Ax \subseteq N$.

Hence $Ax\otimes B \subseteq N\otimes B$ by flatness of $B$.

$\overset{0}{\not=}$ ⇐ by faithfully flat (Thm 3.31)

Then $Ax\otimes B = (x\otimes 1)B$, therefore $x\otimes 1 \neq 0$ ~~by Thm 3.31~~

(2) By base change, $B\otimes_A A/I = B/IB$ is f.f over $A/I$

By (1) $\Rightarrow IB\cap A = I$. (因 $A/I \to B/IB$ injective).

(3). Let $\mathfrak{p} \in \operatorname{Spec} A$. The ring $B_\mathfrak{p} = B\otimes A_\mathfrak{p}$ is f.f over $A_\mathfrak{p}$.

Hence $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. Take a maximal ideal $\mathfrak{m} \subseteq B_\mathfrak{p}$, which contains $\mathfrak{p}B_\mathfrak{p}$.

Then $\mathfrak{m}\cap A_\mathfrak{p} \supseteq \mathfrak{p}A_\mathfrak{p}$, therefore $\mathfrak{m}\cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ since $\mathfrak{p}A_\mathfrak{p}$ maximal

Put $\mathfrak{P} = \mathfrak{m}\cap B$, we get $\mathfrak{P}\cap A = (\mathfrak{m}\cap B)\cap A = \mathfrak{m}\cap A = (\mathfrak{m}\cap A_\mathfrak{p})\cap A \overset{= \mathfrak{p}A_\mathfrak{p}\cap A}{=} \mathfrak{p}$ $\blacksquare$

**Thm 3.35** $\varphi : A \to B$ homo. of rings. TFCAE:

(1) $\varphi$ is faithfully flat.

(2) $\varphi$ is flat, and ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective

(3) $\varphi$ is flat and for any maximal ideal $\mathfrak{m} \subseteq A$, there exists a maximal ideal $\mathfrak{m}' \subseteq B$ lying over $\mathfrak{m}$.

**proof** (1) $\Rightarrow$ (2) by 3.34.

(2) $\Rightarrow$ (3). $\exists \mathfrak{p}' \in \operatorname{Spec} B$ with $\mathfrak{p}' \cap A = \mathfrak{m}$.

If $\mathfrak{m}'$ is any maximal ideal of $B$ containg $\mathfrak{p}'$, we ha

$\mathfrak{m}' \cap A = \mathfrak{m}$ as $\mathfrak{m}$ is maximal.

(3) $\Rightarrow$ (1). the existence of $\mathfrak{m}'$ implies $\mathfrak{m}B \neq B$.

by Thm 3.31 $\Rightarrow$ $B$ is f-f over $A$.

f-f descent

**Prop 3.36** $A$: ring, $B$: f.f $A$-algebra. $M$: $A$-module.

then (1) $M$ is flat (resp. f-f) over $A$ $\Leftrightarrow$ $M \otimes_A B$ is so over $B$.

(2) when $A$ is local and $M$ is finite over $A$, we have

$M$ is free $\Leftrightarrow$ $M \otimes_A B$ is $B$-free.

**proof** (1) $\Rightarrow$ clear

$\Leftarrow$ follows from the fact that: for any seq $S$ of $A$-module, we have $(S \otimes_A M) \otimes_A B = (S \otimes_A B) \otimes_B (M \otimes_A B)$

(2) $\Rightarrow$ trivial

$\Leftarrow$ freenes of $M$ is equivalent to flatnes. Then apply (1).

**Thm 3.37** (Going-down for flat morphism) 以后处理 going-up.

$\phi : A \to B$ flat morphism of rings. Then the going-down theorem holds for $\phi$, i.e., for any $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec} A$ s.t. $\mathfrak{p} \subseteq \mathfrak{p}'$, and for any $Q' \in \operatorname{Spec} B$ lying over $\mathfrak{p}'$,

(GD)
$$\begin{tikzcd}
 & Q' \arrow[d, mapsto] & \in \operatorname{Spec}B \arrow[d] \\
\mathfrak{p} \subseteq & \mathfrak{p}' & \text{in } \operatorname{Spec}A
\end{tikzcd}$$

there exist $Q \in \operatorname{Spec} B$ lying over $\mathfrak{p}$ such that $Q \subseteq Q'$

(可推广到 n个 prime tower的情况)

**pf** Let $Q', \mathfrak{p}', \mathfrak{p}$ as in (GD)

$B_{Q'}$ is flat over $A_{\mathfrak{p}'} \Rightarrow A_{\mathfrak{p}'} \xrightarrow{\text{local rings}} B_{Q'}$ is f.f.

$\Rightarrow \operatorname{Spec} B_{Q'} \to \operatorname{Spec} A_{\mathfrak{p}'}$ surjective

Let $Q^*$ be a prime ideal of $B_{Q'}$ lying over $\mathfrak{p}A_{\mathfrak{p}'}$. Then $Q = Q^* \cap B$ is a prime ideal of $B$ lying over $\mathfrak{p}$ and contained in $Q'$. ▦

## §4 Chain conditions, Noether rings and Artin rings

$\Sigma$: partially ordered set with order $\leq$, which is reflexive, transitive and such that $x \leq y \wedge y \leq x \Rightarrow x = y$.

**Prop 4.1** 以下条件对$\Sigma$等价：

(1) Every increasing sequence $x_1 \leq x_2 \leq \cdots$ in $\Sigma$ is stationary, i.e., there exists $n$ such that $x_n = x_{n+1} = \cdots$.

(2) Every non-empty subset of $\Sigma$ has a maximal element.

**proof** (2) ⇒ (1). The set $\{x_m\}_{m \geq 1}$ has a maximal element, say $x_n$. Then

$$x_n = x_{n+1} = x_{n+2} = \cdots.$$

(1) ⇒ (2) If (2) flase, there is a non-empty $\overset{\text{sub}}{\text{set}}$ ~~$T\subseteq\Sigma$~~ $T\subseteq\Sigma$ with no maximal element, and we can construct inductively a non-termi strictly increasing seq in $T$.

a.c.c = maximal condition for a partially ordered set.

d.c.c = minimal condition for a partially ordered set.

Definition 4.2 $X\in \text{Top}$. We say $X$ is Noether if every non-empty or family of open subsets of $X$ has a maximal element, or equivalently, every non family of closed subsets of $X$ has a minimal element.

$$\text{Noeter} \Leftrightarrow \{\text{open of } X\}\in \text{a.c.c} \Leftrightarrow \left\{\begin{matrix}\text{closed}\\ \text{of } X\end{matrix}\right\}\in \text{d.c.c}$$

We say $X$ is locally Noetherian if each point $x\in X$ has a neighbor which is a Noether space.

4.3 Noether induction method

Let $E$ be an ordered set which satisfies minimal conditions. (比如取 $E=\{$某个No 空间

$P$: a property about elements of $E$ suchthat:

for all $a\in E$, if $P(x)$ holds for all $x<a$, then $P(a)$ holds.

Then $P(x)$ holds for all $x\in E$.

Pf In fact, let $F=\{x\in E \mid P(x) \text{ not hold}\}$. 若$F\neq\phi$. Then $F$ has a minimal elemen $a\in E$. Then for $x<a$, $P(x)$ holds. By assumption on $P \Rightarrow P(a$ 矛盾

Prop 4.4 (1) Subspace of a Noether space is Noether.

If X is a finite union of Noether spaces, then X is Noether.

(2) a Noether space is quasi-compact $\left( 如果\ X = \bigcup_{\substack{\text{open} \\ i\in I}} X_i \Rightarrow 可取有限个\ i \in I,\ X_i \text{ cover } X \right)$

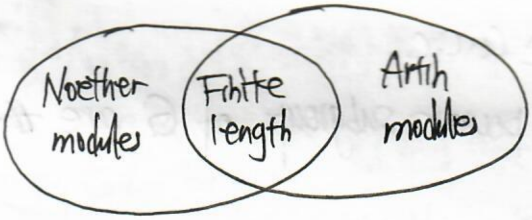
If any open subset of a space is quasi-compact, then X is Noether.

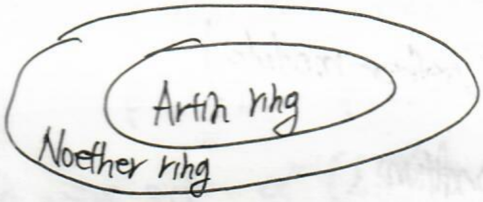
$\left( \{U_i \mid U_i \underset{\text{open}}{\subseteq} X\}_{i\in I},\ U = \bigcup_{i\in I} U_i \text{ open, hence quasi-compact} \Rightarrow \bigcup_{i\in I} U_i \overset{\exists}{=} U_{i_0} \cup \cdots \cup U_{i_n} \right)$

(3) By Noether induction 4.3, a Noether space has only finitely many irr. components.

Apply this to study rings (and modules)

Note that closed subsets of Spec A are of the form Spec A/I for some ideals I.





ascending chain condition

a.c.c = maximal condition for a partially ordered set.

d.c.c = minimal condition for a partially ordered set.

Definition 4.5 (1) A: Noether ring. $\Leftrightarrow$ $\left\{ \begin{matrix} \text{ideals of} \\ A \end{matrix} \right\}$ with inclusion order satisfies a.c.c

~~$\Leftrightarrow$ {closed sub.~~

A: Artin ring $\Leftrightarrow$ $\left\{ \begin{matrix} \text{ideal of} \\ A \end{matrix} \right\}$ satisfies d.c.c.

(2) $M \in \mathrm{Mod}_A$. $M$ is a Noether $A$-module $\overset{\text{def}}{\Longleftrightarrow}$ $\left\{ \begin{matrix} \text{submodule} \\ \text{of } M \end{matrix} \right\}$ with ordered by $\subseteq$ satisfies a.c.c.

Same to define Artin module.

Note that (1) is a special case of (2) if regard $A$ as a $A$-module.

2024.04.03

Example 4.6 (1) Finite ab. groups (as $\mathbb{Z}$-module) satisfies both a.c.c and d.c.c

(2) $\mathbb{Z}$ satisfies a.c.c, but not d.c.c (e.g. $(p) \supsetneqq (p^2) \supsetneqq (p^3) \supsetneqq \cdots$)

(3) $p$: prime

$G = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \{x \in \mathbb{Q}/\mathbb{Z} \mid x \text{ has order a power of } p\}$

For each $n \geq 0$, $G$ has exactly one subgroup $G_n = \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ of order $p^n$

$\Rightarrow G_0 \subsetneqq G_1 \subsetneqq \cdots \subsetneqq G_n \subsetneqq \cdots$

$\Rightarrow G$ does not satisfy the a.c.c.

On the other hand, the only power subgroups of $G$ are the $G_n$

$\Rightarrow G$ satisfies d.c.c.

($G$ is Artin module, but a not Noether module)

(4) $\mathbb{Z}[\frac{1}{p}]$ satisfies neither chain conditions:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}[\frac{1}{p}] \arrow[r] & G \arrow[r] & 0 \\ & \text{not d.c.c} & & \text{not a.c.c} & \end{tikzcd}$$

$\mathbb{Z} \notin \text{d.c.c} \Rightarrow \mathbb{Z}[\frac{1}{p}] \notin \text{d.c.c}$

$G \notin \text{a.c.c} \Rightarrow \mathbb{Z}[\frac{1}{p}] \notin \text{a.c.c.}$

(5) $k$: field. $k[x]$: Noether ring. ($(f) \subseteq (g) \Leftrightarrow g \mid f$)

but not Artin, since $(f) \supsetneqq (f^2) \supsetneqq \cdots$ 与$\mathbb{Z}$类似

(6) $k[X_1,\cdots,X_n,\cdots]$ polynomial ring in ∞-determinates.

$(X_1) \subsetneqq (X_1,X_2) \subsetneqq \cdots\cdots$ Not Noether

$(X_1) \supsetneqq (X_1^2) \supsetneqq (X_1^3) \supsetneqq \cdots\cdots$ Not Artin. ▦

**Prop 4.7** $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ exact seq of A-modules.

**then** (1) M is Noether ⇔ M' and M'' are Noetherian.

(2) M is Artin ⇔ M' and M'' are Artin.

**Corollary 4.8** If $M_i (1 \le i \le n)$ are Noether (resp. Artin) A-modules, then so is $\bigoplus_{i=1}^{n} M_i$.

**Pf 4.7** Only prove (1). (2) is Similar.

"⇒" M'的子模升链也是M的子模升链
$\beta^{-1}$(M''的子模升链)是M的子模升链 } ⇒ stationary.

"⇐" $L_1 \subseteq L_2 \subseteq \cdots\cdots$ M中子模升链.

then $\alpha^{-1}(L_1) \subseteq \alpha^{-1}(L_2) \subseteq \cdots$
$\beta(L_1) \subseteq \beta(L_2) \subseteq \cdots$ are stationary.

Choose $n \gg 0$ s.t $\alpha^{-1}(L_n) = \alpha^{-1}(L_{n+1}) = \cdots$
$\beta(L_n) = \beta(L_{n+1}) = \cdots\cdots$

Then $L_n = L_{n+1} = \cdots\cdots$ ▦

**Prop 4.9** M is a Noether A-module ⇔ Every submodule of M is f.g.

In particular, a ring A is Noether ring ⇔ every ideal of A is f.g.

**proof** "$\Leftarrow$" $M_1 \subseteq M_2 \subseteq$ submodules of $M$.

$N = \bigcup M_i$ submodule of $M$, which is f.g by $x_1, \dots, x_r$.

$\Rightarrow \exists\, n$ s.t $x_1, \dots, x_r \in M_n \Rightarrow M_n = M_{n+1} = \cdots = N$.

"$\Rightarrow$" $N \subseteq M$ submodule.

Show: $N$ is f.g. If not, can choose $N_1 \subsetneq N_2 \subsetneq \cdots$

$\Sigma = \{$f.g. submodules of $N\}$. Noether条件

$0 \in \Sigma \Rightarrow \Sigma$ is not empty $\Rightarrow \Sigma$ has a maximal element $N_0$

若 $N \neq N_0$, choose $x \in N$, $x \notin N_0$, then $N_0 \subsetneq N_0 + A \cdot x \in \Sigma$

与 $N_0$ 极大矛盾!

**Prop 4.10** A: Noether ring (resp. Artin ring)

M: f.g. A-module.

Then M is Noether (resp. Artin) as A-module.

**proof** choose $A^N \twoheadrightarrow M \to 0$. $A^N$ is Noether (resp. Artin) by 4.8.

By 4.9 $\Rightarrow \operatorname{Ker}(A^N \twoheadrightarrow M)$ is Noether (resp. Artin).

$\underset{K}{\overset{\text{def}}{\parallel}}$ ($K$ submodule of $A^N$)

By 4.8 $\Rightarrow M = \operatorname{Coker}(K \to A^N)$ is Noether (resp. Artin).

$$0 \to K \to \underset{\text{Noether (resp. Artin).}}{\underset{\uparrow}{A^N}} \to M \to 0.$$

**Def 4.11** ① $M \in \mathrm{Mod}_A$. A chain of submodule of M is a sequ

$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = (0))$.

the length of this chain is $n$ ("$\supsetneq$" 的个数)

② A composition series of M is a maximal chain, such that one cannot insert any extra submodules, or equivalently each quotient $M_{i-1}/M_i (1\le i\le n)$ is simple (子模只有0与自身).

**Prop 4.12** Suppose that $M$ has a composition series of length $n$. Then every composition series of $M$ has length $n$, and every chain in $M$ can be extended to a composition series.

(类似 Jordan-Hölder 定理 thm for finite groups: $M$的任何两个 composition series 有相同的 quotients $\{M_{i-1}/M_i\}_{1\le i\le n}$)
$\nearrow$ 不计次序.

**proof** define $\ell(M)$ = least length of composition series of $M$

$\ell(M)=\infty$ if $M$ has no composition series.

**Claim 1** For $N \subsetneqq M$, we have $\ell(N) < \ell(M)$.

Let $(M_i)$ be a composition series of $M$ of minimum length.

Consider $N_i = N \cap M_i \subseteq N$. Since $N_{i-1}/N_i \subseteq M_{i-1}/M_i$ $\swarrow$ simple.

$$\Rightarrow N_{i-1}/N_i = M_{i-1}/M_i \text{ or } N_{i-1}/N_i = 0.$$

removing repeated terms in $N_0 \supseteq N_1 \supseteq \cdots$, get a composition series $N$, thus $\ell(N) \le \ell(M)$.

若 $\ell(N)=\ell(M)=n$, then $N_{i-1}/N_i = M_{i-1}/M_i$ for each $1\le i\le n$

$$\Rightarrow M_{n-1} = N_{n-1} \Rightarrow M_{n-2} = N_{n-2} \Rightarrow \cdots \text{ finally } M=N.$$

**Claim 2** Any chain in $M$ has length $\le \ell(M)$.

Let $M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_k$ be a chain of length $k$. Then

by claim 1 $\Rightarrow \ell(M) > \ell(M_1) > \cdots > \ell(M_k) = 0 \Rightarrow \ell(M) \ge k$.

Now consider any composition series $(M_i)$ of $M$. by def of $\ell(M)$ 最小 $\Rightarrow (M_i)$的长度 $\ge \ell(M)$.
由 claim 2 $\Rightarrow (M_i)$的长度 $\le \ell(M)$. Thus $(M_i)$的长度为 $\ell(M)$.
从而 $M$的任何 composition series is of same length.

For any chain, if it has length $\ell(M)$, then by claim 2, it must be a composition series.

If its length $< \ell(M)$, it is not a composition series, hence not maximal, and therefore new terms can be inserted until its length is $\ell(M)$.

**Prop 4.13** $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ exact seq of finite length.

then $\ell(M) = \ell(M') + \ell(M'')$

**proof** $M'$的合成列 与 $\beta^{-1}(M''$的一个合成列$)$ 合成 $M$的一个合成列. □

**Exercise 4.14** $V/k$ vector space over a field $k$. 以下条件等价:

(1) $\dim V < \infty$. (2) $\ell(V) < \infty$. (3) a.c.c (4) d.c.c.

If these conditions are satisfied, then $\dim V = \ell(V)$.

Ⓟⓕ $(1) \Rightarrow (2) \Rightarrow (3)$ & $(4)$ clear.

show $(3) \Rightarrow (1)$ (and $(4) \Rightarrow (1)$).

If (1) not hold, then $\exists$ infinite sequence $(x_n)_{n\geq 1}$ of linearly indep elements

then $\mathbb{C}x_1 \subsetneq \mathbb{C}x_1 \oplus \mathbb{C}x_2 \subsetneq \cdots\cdots$ 矛盾.

$V_n = \mathbb{C}\{x_{n+1}, x_{n+2}, \cdots\cdots\}$

$V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \cdots\cdots$ 矛盾. ■

**Corollary 4.15** $A$: ring with $(0) = m_1 \cdots m_n$ ( $m_i \in \max(A)$. $m_i$ may not necessarily distinct

then $A$ is Noether iff $A$ is Artin

(之后用于证明: Artin = Noether + dim 0)

proof Consider the chain of ideals $A \supsetneq m_1 \supsetneq m_1 m_2 \supsetneq \cdots \supsetneq m_1 \cdots m_n = 0$.

each $m_1 \cdots m_{i+1} / m_1 \cdots m_i$ is a vector space over the field $A/m_i$

hence a.c.c $\Leftrightarrow$ d.c.c for each factor.

By Prop 4.7, a.c.c (resp. d.c.c) for each factor $\Leftrightarrow$ a.c.c (resp. d.c.c) for A

hence a.c.c $\Leftrightarrow$ d.c.c. for A. ∎

## Noether 性质在商与 localization 下稳定

Prop 4.16 A : Noether ring.

(1) $A \twoheadrightarrow B$ surj homo. Then B is Noether.

proof B 中 ideal 对应于 A 中包含 $\mathrm{Ker}(A \twoheadrightarrow B)$ 的 ideal.

(2) $S \subseteq A$ any multi closed subset. Then $S^{-1}A$ Noether.

In particular, for any $\mathfrak{p} \in \mathrm{Spec}A$, and any $0 \neq f \in A$ ($f$ non unit and non-zero divisor)

$A_{\mathfrak{p}}$ and $A_f$ are Noether.

证明 $$\left\{\begin{matrix}\text{ideals of}\\ S^{-1}A\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{contracted}\\ \text{ideals of}\\ A\end{matrix}\right\}$$ preserving order.

(3) $A \subseteq B$ ring. B is a f.g A-module (B is a finite A-algebra)

Then B is a Noether ring.

Pf By Prop 4.10 $\Rightarrow$ B is a Noether A-module, hence also Noether B-module. ∎

**Example 4.17** 数论中的例子.

$B = \mathbb{Z}[i] = \mathbb{Z}[X]/(X^2+1)$ Noether ring.

Any ring of integers in any alg. number fields (finite ext of $\mathbb{Q}$) is a No

ring.

**Thm 4.18 (Hilbert's basis theorem)**

If $A$ is a Noetherian ring, then $A[X]$ is a Noetherian ring.

Hence $A[X_1, \cdots, X_n]$ is also Noether ring and any f.g. A-algebra

also Noetherian.

As any field $k$ is Noether $\Rightarrow \dfrac{k[X_1, \cdots, X_n]}{I}$ is Noether.

Any f.g. ring over $\mathbb{Z}$ and every f.g. $k$-algebra over a field $k$ is

Noether.

(以后会证明 $A[[X]] = \left\{\begin{matrix}\text{formal power} \\ \text{series in } A\end{matrix}\right\}$ is Noether)

(pf) $I \subseteq A[X]$ ideal. By Prop 4.9, 只要证明: $I$ is f.g.

$I_0 = \left\{\begin{matrix}I\text{中多项式} \\ \text{的首项系数}\end{matrix}\right\}$. $I_0$ is an ideal of $A$. Hence f.g as A

Say $I_0 = (a_1, \cdots, a_n) \subseteq A$.

For $1 \le i \le n$, 设 $a_i$ 是 $f_i \in A[X]$ 的首项: $f_i = a_i x^{r_i}$ + lower terms

$r = \max\limits_{1 \le i \le n} r_i$, $\{f_i\}$ generate an ideal $I' \subseteq I$ in $A[X]$.

**Claim** $I$中元素可写成 $g+h$ $(\deg g < r,\ h \in I')$.

Indeed, let $f = ax^m + (\text{lower terms})$, $f \in I$.

We have $a \in I_0$, if $m \geq r$, write $a = \sum_{i=1}^{n} u_i a_i$, $u_i \in A$

then $f - \sum u_i a_i x^{m-r_i} \in I$ has degree $< m$.

Proceeding in this way, we can go on subtracting elements of $I' \subseteq I$ from $f$, until we get a polynomial $g$ of degree $< r$, i.e, $f = g + h$ ($\deg g < r$, $h \in I'$).

Let $M$ be the $A$-module generated by $1, x, \dots, x^{r-1}$, then we have proved that $I = (I \cap M) + I'$.

But $M$ is f.g $A$-module $\Rightarrow$ $M$ is Noether

$\Rightarrow I \cap M$ is f.g as an $A$-module

$\Rightarrow I = I \cap M + I'$ is f.g. ▣

4.19 Weak version of Hilbert's Nullstellensatz

$A$ = f.g. $k$-algebra. $m \subseteq A$ maximal ideal. then $A/m$ is a finite alg ext of $k$. 特别若 $\bar{k} = k$, then $A/m = k$.

this is a corollary of the following property (放在局部化之章节证)

$k$ = field. $E$ = f.g. $k$-algebra.

If $E$ is a field, then it is a finite alg. ext of $k$.

现讨论 Artin ring (ring which satisfies d.c.c on ideals)

$\uparrow$

仅仅改acc却稍复杂.

Prop 4.20 $A$: Artin ring. 2024.04.08

(1) Every prime ideal $\mathfrak{p}$ is maximal (从而 nilradical of $A$ equals the Jacobson radical of $A$)

(2) $A$ has only a finite number of prime/maximal ideals
(Spec $A$ is a finite space)

(3) The nilradical $\mathfrak{N}$ of $A$ is nilpotent.

proof (1) $B = A/\mathfrak{p}$ integral domain and Artin ring.

We show $B$ is a field.

Let $x \in B \backslash \{0\}$. We show $x$ has an inverse.

By d.c.c $\Rightarrow \exists\, n \geq 0$ s.t $(x^n) = (x^{n+1})$.

$\Rightarrow x^n = x^{n+1}y$ for some $y \in B$.

But $B$ is integral $\Rightarrow 1 = xy \Rightarrow B$ is a field $\Rightarrow \mathfrak{p}$ is maxi

(2) $\{\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r \mid \mathfrak{m}_i\text{: maximal}\}$ has a minimal element, say $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$

Then for any maximal $\mathfrak{m}$, we have

$$\mathfrak{m} \cap (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \quad (\text{thus } \subseteq \mathfrak{m}$$

By prime avoidance lemma $\overset{1.33}{\Rightarrow} \mathfrak{m} \supseteq \mathfrak{m}_i$ for some $i$.

But $\mathfrak{m}$ and $\mathfrak{m}_i$ are maximal $\Rightarrow \mathfrak{m} = \mathfrak{m}_i$.

(3). By d.c.c, we have $\mathfrak{N}^k = \mathfrak{N}^{k+1} = \cdots =: I$ for some $k > 0$

下证明 $I = 0$.

若 $I \neq 0$, 则 $\Sigma = \{J \subseteq A \text{ ideal} \mid \overset{IJ \neq 0}{[illegible]} \neq 0\}$ 非空 $(I \in \Sigma)$, 从

有极小元 $C\in\Sigma$ $(C\neq 0)$、下证 $C$ 由一个元素生成.

Let $0\neq x\in C$ s.t $xI\neq 0$.

Then $(x)\in\Sigma$ and $(x)\subseteq C$ $\xRightarrow[C\text{极小}]{}$ $C=(x)$.

But $(xI)\cdot I=xI^2=xI\neq 0$ and $xI\subseteq(x)$

hence $xI=(x)$ by minimality of $C=(x)$.

hence $x=xy$ for some $y\in I$.

$\Rightarrow x=xy=xy^2=\cdots=xy^n=\cdots$

But $y\in I=n^k$ $\Rightarrow y$ is nilpotent $\Rightarrow x=xy^N=0$ $N\gg 0$.

This contradicts of the choice of $x$ $\Rightarrow I=0$. ▦

<u>Def 4.21 (Krull dimension)</u> $\dim A=\sup\left\{n \,\middle|\, \begin{array}{l}\text{存在长度为 } n \text{ 的 chain of prime}\\ \text{ideals } \mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n\end{array}\right\}$

$\dim A\geq 0$ or $\dim A=+\infty$.

$\dim(\text{域})=0$, $\dim\mathbb{Z}=1$. 以后证 $\dim k[X_1,\cdots,X_n]=n$.

<u>Thm 4.22</u> $A$: ring. $A$ is Artin $\Leftrightarrow$ $A$ is Noether and $\dim A=0$.

<u>Pf</u> "$\Rightarrow$" Since prime in $A$ are maximal $\Rightarrow\dim A=0$.

<u>下证 $(0)=\mathfrak{m}_1^{\cdot}\cdots\mathfrak{m}_n^{\cdot}$</u> (Then Coro. 4.15 $\Rightarrow$ $A$ is Noether)

Indeed: Let $\mathfrak{m}_i$ $(1\leq i\leq n)$ be the distinct maximal ideals of $A$.

Then $\prod_{i=1}^{n}\mathfrak{m}_i^k\subseteq\left(\bigcap_{i=1}^{n}\mathfrak{m}_i\right)^k=n^k=0$ for $k\gg\infty$

"$\Leftarrow$" 习题. 只要证明 $A$ 只有有限个 maximal prime $\mathfrak{m}_1,\cdots,\mathfrak{m}_n$ 且 $n=\bigcap_{i=0}^{n}\mathfrak{m}_i$ is nilpotent. then by 4.15 $\Rightarrow$ $A$ is Artin. ▦

**Prop 4.23** A: Noether local ring. $\mathfrak{m} \subseteq A$ maximal ideal.

Then exactly one of the following two statements are true:

(1) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all $n$ (A is not Artin)

(2) $\mathfrak{m}^n = 0$ for some $n$, in which case A is an Artin local ring)

**Pf** Suppose $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n$. By Nakayama $\Rightarrow \mathfrak{m}^n = 0$.

Let $\mathfrak{p} \in \operatorname{Spec} A \Rightarrow \mathfrak{m}^n = 0 \subseteq \mathfrak{p} \Rightarrow \mathfrak{m} \underset{\text{avoidence lemma 1.33}}{\subseteq} \mathfrak{p}$ (hence $\mathfrak{m} = \mathfrak{p}$)

Hence $\mathfrak{m}$ is the only prime ideal of A.

By 4.15 $\Rightarrow$ A is Artin. ∎

从一个 Noether ring A 出发, localize $\rightsquigarrow A_\mathfrak{p} \rightsquigarrow A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})^N$ is Artin

Prop 4.23 给出一个 Noether local ring 何时不是 Artin/何时是 Artin.

**Thm 4.24 (Structure thm for Artin rings)**

An Artin ring A is (uniquely up to isom) a finite direct product of Artin local rings.

**proof** Let $\mathfrak{m}_i$ ($1 \le i \le n$) be the distinct maximal ideals of A

By prop 4.20 $\Rightarrow \exists k > 0$ s.t $\prod_{i=1}^{n} \mathfrak{m}_i^k = 0$.

the ideals $\mathfrak{m}_i^k$ are coprime to each other $\Rightarrow \cap \mathfrak{m}_i^k = \prod \mathfrak{m}_i^k$

$$\Rightarrow A \cong A/\bigcap_{i=1}^{n} \mathfrak{m}_i^k \cong \prod_{i=1}^{n} A/\mathfrak{m}_i^k$$

each $A/\mathfrak{m}_i^k$ is an Artin local ring. ∎

**Example 4.25** A : ring with only one prime ideal.

A may not be Noether (hence not artin).

e.g. $A = k[X_1, X_2, \dots, ]$, $I = (X_1, X_1^2, \dots, X_n^n, \dots)$

the ring $A/I$ has only one prime ideal $(\bar{x}_1, \bar{x}_2, \dots)$.

$\Rightarrow A/I$ is local ring of dimension 0.

But $A/I$ is not Noether since $(\bar{x}_1, \bar{x}_2, \dots)$ is not f.g. $\square$

**讨论 4.26** A : local ring with maximal ideal $m$, $k = A/m$ residue field.

The A-module $m/m^2$ is annihilated by $m$, and therefore has the structure of a $k$-vector space.

If $m$ is f.g, the image in $m/m^2$ of a set of generators of $m$ will span $m/m^2$ as a vector space.
$\hookrightarrow (\Rightarrow \dim m/m^2 < \infty)$

**Prop 4.27** A : Artin local ring. TFAE :

(1) Every ideal in A is principal.

(2) the maximal ideal $m$ is principal.

(3) $\dim_k m/m^2 \leq 1$.

**proof** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) clear by 4.26.

(3) $\Rightarrow$ (1) If $\dim_k m/m^2 = 0 \Rightarrow m = m^2 \Rightarrow m = 0$ by Nakayama

$\Rightarrow A/m = A$ is a field $\Rightarrow$ (1).

~~证~~ If $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, then $\mathfrak{m}$ is a principal ideal, say $\mathfrak{m} = (x)$.

Let $(0) \subsetneq I \subsetneq A$ ideal. 下证 $I$ is principal.

We have $\mathfrak{m}$ is nilpotent.

$\Rightarrow \exists r$ s.t. $I \subseteq \mathfrak{m}^r$ but $I \subsetneq \mathfrak{m}^{r+1}$

$\Rightarrow \exists y \in I$ s.t. $y = ax^r$, $y \notin (x^{r+1})$

$\Rightarrow a \notin (x) = \mathfrak{m}$ and $a$ is a unit in $A$.

$\Rightarrow x^r \in I \Rightarrow \mathfrak{m}^r = (x^r) \subseteq I \Rightarrow I = \mathfrak{m}^r = (x^r)$

$\Rightarrow I$ is principal. ∎

Example 4.28 $p \in \operatorname{Spec}\mathbb{Z}$ prime number.

$\mathbb{Z}/p$, $k[x]/(f^n)$ ($f$: irr) satisfies the conditions of 4.27.

But the Artin local ring $k[x^2, x^3]/(x^4)$ does not: $\mathfrak{m}$ is gene

by $x^2, x^3 \pmod{x^4} \Rightarrow \mathfrak{m}^2 = 0$ and $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$. ∎

Another filtration for modules (类比合成列).

We will prove:

Thm 4.29 $A$: Noether ring. $M$: finite $A$-module, $M \neq 0$. Then there is a chain of submodules $(0) = M_0 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} A$ $(1 \le i \le n)$.

(想法：只要找到一个 $M_1 \cong A/\mathfrak{p}_1$, $\mathfrak{p}_1 \in \operatorname{Spec} A$ 即可，然后对 $M/M_1$ 类似

Definition 4.30 $A$: Noether ring, $M \in \mathrm{Mod}_A$. $\mathfrak{p} \in \mathrm{Spec}\,A$. We say $\mathfrak{p}$ is an associated prime of $M$ if one of the following equivalent conditions holds:

(1) $\exists x \in M$ with $\mathrm{Ann}(x) = \mathfrak{p}$.

(2) $M$ contains a submodule isomorphic to $A/\mathfrak{p}$

Ⓟⓕ (1) $\Rightarrow$ (2) $A.x \cong A/\mathfrak{p}$, $\mathrm{Ker}(A \to Ax) = \mathfrak{p}$.

(2) $\Rightarrow$ (1) clear.

Put $\mathrm{Ass}_A(M) = \mathrm{Ass}(M) =$ set of associated primes of $M$.

Prop 4.31 Let $\mathfrak{p}$ be a maximal element of $\{\mathrm{Ann}(x) \mid x \neq 0 \text{ in } M\} = \Sigma$

Then $\mathfrak{p} \in \mathrm{Ass}(M)$.

In particular, $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} = \bigcup_{x \in M\setminus\{0\}} \mathrm{Ann}(x)$ = set of zero divisors of $M$.

$\mathrm{Ass}(M) = \phi \Leftrightarrow M = 0$.

$M \neq 0 \Rightarrow \mathrm{Ass}(M) \neq \phi$.

proof We show $\mathfrak{p}$ is a prime ideal. Assume $\mathfrak{p} = \mathrm{Ann}(x)$, $ab \in \mathfrak{p}$, $b \notin \mathfrak{p}$.

Then $bx \neq 0$ and $abx = 0 \Rightarrow a \in \mathrm{Ann}(bx)$. but $\mathfrak{p} = \mathrm{Ann}(x) \subseteq \mathrm{Ann}(bx)$

$\Uparrow$ maximal in $\Sigma$

Thus $\mathrm{Ann}(x) = \mathrm{Ann}(bx) = \mathfrak{p} \Rightarrow a \in \mathfrak{p}$. ▦

proof of Thm 4.29 $M \neq 0$. Can choose $M_1 \subseteq M$ such that $M_1 \cong A/\mathfrak{p}_1$ for some $\mathfrak{p}_1 \in \mathrm{Ass}(M) \neq \phi$.

If $M_1 \neq M$, then apply the same argument to $M/M_1$, can find $M_2$ and so on.

But $M$ is a Noether $A$-module $\Rightarrow$ the process must stop in finite steps. ▦

Lemma 4.32 If $0 \to M' \to M \to M''\cancel{\to 0}$ is an exact seq of $A$-modules, then

$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.

(proof) Take $\mathfrak{p} \in \mathrm{Ass}(M)$, choose $N \subseteq M$ isomorphic to $A/\mathfrak{p}$.

If $N \cap M' = (0)$, then $N$ is isomorphic to a submodule of $M'' \Rightarrow \mathfrak{p} \in \mathrm{Ass}(M'')$.

If $N \cap M' \neq (0)$, pick $0 \neq x \in N \cap M'$. Since $N \cong A/\mathfrak{p}$ and $A/\mathfrak{p}$ is a domain, We have $\mathrm{Ann}(x) = \mathfrak{p} \Rightarrow \mathfrak{p} \in \mathrm{Ass}(M')$.

<u>Lemma 4.33</u> A: Noether ring. M: finite A-module.
Then $\mathrm{Ass}(M)$ is a finite set.

<u>proof</u> By 4.29 and 4.32 $\Rightarrow \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \cdots \cup \mathrm{Ass}(M_n$

But $\mathrm{Ass}(M_i/M_{i-1}) = \mathrm{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$

$\Rightarrow \mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

<u>Lemma 4.34</u> A: Noether. $S \subseteq A$ multiclosed. $f: \mathrm{Spec}\, S^{-1}A \to \mathrm{Spec}\, A$.

$M \in \mathrm{Mod}_A$. Then $\mathrm{Ass}_A(S^{-1}M) = f(\mathrm{Ass}_{S^{-1}A}(S^{-1}M))$

$$= \mathrm{Ass}_A(M) \cap \{\mathfrak{p} \mid \mathfrak{p} \cap S = \phi\}$$

(use any ideal of A is f.g).

<u>Thm 4.35</u> A: Noether, $M \in \mathrm{Mod}_A$. Then $\mathrm{Ass}(M) \subseteq \mathrm{Supp}\, M = \{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0\}$

Any minimal element of $\mathrm{Supp}\, M$ is in $\mathrm{Ass}(M)$.

<u>proof</u> $\forall \mathfrak{p} \in \mathrm{Ass}(M)$, $\exists\ 0 \to A/\mathfrak{p} \to M$ exact

$\Rightarrow 0 \to \underset{\substack{\nparallel \\ \phi}}{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} \to M_{\mathfrak{p}}$ exact. $\Rightarrow M_{\mathfrak{p}} \neq 0$
$\Rightarrow \mathfrak{p} \in \mathrm{Supp}(M)$.

Now let $\mathfrak{p} \in \mathrm{Supp}(M)$ be a minimal element.

By lemma 4.34, $\mathfrak{p} \in \mathrm{Ass}(M) \Leftrightarrow \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$

Therefore replacing A and M by $A_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$, we can assume that $(A, \mathfrak{p})$ is a local ring with $M \neq 0$, and that $M_{\mathfrak{q}} = 0$ for a prime $\mathfrak{q} \subsetneq \mathfrak{p}$.

Thus $\mathrm{Supp}(M) = \{\mathfrak{p}\}$.

Since $\mathrm{Ass}(M)$ is not empty, and $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M) \Rightarrow \mathfrak{p} \in \overset{\mathrm{Ass}(M)}{\cancel{\mathrm{Ass}(M)}}$.

**Corollary 4.36** $I \subseteq A$ ideal. Then the minimal associated primes of the A-module $A/I$ are precisely the minimal prime over ideals of $I$.

**proof.** $\mathrm{Ass}(A/I) \subseteq \mathrm{Supp}(A/I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$. ∎