DELOCALIZATION OF SCHRÖDINGER EIGENFUNCTIONS

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Abstract. A hundred years ago, Einstein wondered about quantization conditions for classically ergodic systems. Although a mathematical description of the spectrum of Schrödinger operators associated to ergodic classical dynamics is still completely missing, a lot of progress has been made on the delocalization of the associated eigenfunctions.

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1. Some history

One can date the birth of quantum mechanics back to Planck's 1900 paper [93], when he realized that the statistical model leading to the spectrum of the "black body" had to be discrete, not continuous. To that effect, he introduced the "Planck constant" h, but this was for him a mathematical artefact, without physical foundation. It was Einstein who gave this notion a physical meaning, introducing the idea of quantum of energy in the exchange of energy between electromagnetic field and matter (later called photon) [49]. The amount of energy that can be exchanged between light of frequency ν and matter is discrete, "quantized", it must be an integer multiple of $h\nu$.

This idea was applied by Bohr in 1913 to the planetary model of the atom [28]. Trying to explain the discrete emission/absorption spectrum of the hydrogen, he used the Rutherford model where the electron gravitates around the nucleus submitted to Coulomb attraction, and postulated the quantization of the kinetic momentum : it must be an integer multiple of h. This in turn implied that the energy can only take a discrete set of values, that fitted perfectly well with the experimental spectrum. However, setting up quantization rules for larger atoms turned out to be an inextricable task.

In 1917, Einstein wrote a theoretical paper with an aim to extend the quantization rules to systems with higher degrees of freedom [50]. He modified some rules given earlier by Epstein and Sommerfeld, and he noted that his new rules only made sense if (using modern vocabulary) the system is completely integrable : that is, if there exist some action/angle canonical coordinates, such that the actions are invariants of motion (Einstein's quantization rule is that the values taken by the action variables have to be integer multiples of h). At the end of Einstein's paper, there is a sentence that looks incidental, but may be considered to be the starting point of a whole field of research : *"on the other hand, classical statistical mechanics is essentially only concerned with Type b) [i.e. non integrable systems], for in this case the microcanonical average is the same as the time average"*. The equivalence of time average with the average over phase space is the property called "ergodicity". Einstein's point is the following : if a classical dynamical system is ergodic, the quantization rules do not apply, so how can we describe its spectrum ?

Facing the failure to find quantization rules even for an atom as simple as the helium, Heisenberg set up in 1925 entirely new rules of mechanics [64]. One should work only with observable quantities such as the position or the momentum (but for instance, the trajectory of an electron is not observable); and these "observables" are modelled by matrices (operators), subject to certain commutation rules. The momentum observable p and the position observable q must satisfy $qp - pq = i\hbar I$, where $\hbar$ is the reduced Planck constant $h/2\pi$. Time evolution is governed by the energy observable H; Heisenberg gives a recipe to build the operator H starting from the classical expression of energy. Any other observable A evolves according to the linear equation $i\hbar\frac{dA}{dt} = [A, H]$, where $[\cdot,\cdot]$ stands for the commutator of two operators. The physical spectrum of the system (emitted or absorbed energies) is given by the differences $E_n - E_m$, where (E_n) are the eigenvalues of H.

At the same time, a concurrent theory emerged. In 1923, De Broglie had formulated the idea of *wave mechanics* : in the same way as light, considered to be a wave, was discovered to have a discrete behaviour embodied by the photons, one could do the reverse operation with the particles composing matter, and consider them to be waves as well. In 1926 Schrödinger proposed an evolution equation for a wave/particle of mass m evolving in a force field coming from a potential V [99, 100] :

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \tag{1}$$

where Δ is the Laplacian, and where $\psi = \psi(t, x)$ is a function of time t and of the position $x \in \mathbb{R}^3$ of the particle, called the *wave function*.

The linear partial differential equation (1) can be solved by diagonalizing the differential operator $H = -\frac{\hbar^2}{2m}\Delta + V$. Assume, for instance, we can find an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^3)$ consisting of functions ϕ_n satisfying $H\phi_n = E_n\phi_n$ with $E_n \in \mathbb{R}$. Then the general solution of (1) is

$$\psi(t, x) = \sum_n c_n\phi_n(x)e^{-itE_n/\hbar}$$

where the coefficients $c_n \in \mathbb{C}$ are given by the initial condition at $t = 0$. The physical spectrum is again given by the differences $E_n - E_m$.

Both the Heisenberg and the Schrödinger theories yielded exact results for the hydrogen atoms, but also for larger ones. In fact, they can be shown to be mathematically equivalent. But, as Schrödinger wrote it [101], mathematical equivalence is not the same as physical equivalence. The wave function ψ is absent from Heisenberg's theory. Soon afterwards, Born gave a probabilistic interpretation of the function ψ : $|\psi(x, t)|^2$ represents the probability, in a measurement, to find a particle at position x, at time t. This was in complete disagreement with Schrödinger's intuition, but this is the interpretation that has been retained.

After 1925, Einstein's question may be reformulated as follows : if a classical system is ergodic, and if H is the energy operator governing the system from the point of view of quantum mechanics, what are the patterns exhibited by the eigenvalues of the operator H ? How is classical ergodicity transferred to the quantum system ?

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One may broaden the question by asking about the properties of the wave functions, that is, the eigenfunctions of H (solutions of $H\phi = E\phi$, $E \in \mathbb{R}$), or more generally the solutions $\psi(x,t)$ of the time-dependent solutions of (1). How are the probability densities $|\psi|^2$ localized in space ?

In the mid-fifties, Wigner introduced Random Matrix Theory to deal with the scattering spectrum of heavy nuclei. Although there is no doubt about the validity of the Schrödinger equation, it seems impossible to effectively work with it, in view of the high number of degrees of freedom of such systems. Wigner's hypothesis was that the spectrum of heavy nuclei resembles, statistically, that of certain *ensembles* of large random matrices (the Gaussian Orthogonal Ensemble or the Gaussian Unitary Ensemble). This turns out to fit the experimental data extraordinarily well (pictures may be found in Bohigas' paper [27]).

Unexpectedly, the spectral statistics of Random Matrix Theory were discovered to also fit extremely well with the spectra of certain Schrödinger operators with very few degrees of freedom : the hydrogen atom in a strong magnetic field, as well as some 2-dimensional billiards (in the latter case, the Schrödinger operator is just the Laplacian in a bounded open set of $\mathbb{R}^2$, with Dirichlet boundary condition). See Delande's paper [43] for illustrations. The common point of all these examples is that the underlying classical dynamical system is ergodic, or even *chaotic*, meaning a very strong sensitivity to initial conditions. So, it seems that the answer to Einstein's question could be that : if the classical dynamics is ergodic, or sufficiently chaotic, then the spectrum of the corresponding Schrödinger operator looks like that of a large random matrix. This is known as the Bohigas–Giannoni-Schmit conjecture [26]. However, :

- there is to this day no mathematical proof of this fact; the question may be considered fully open, except for the heuristic arguments given by Sieber and Richter [103], that seem impossible to make mathematically rigourous;
- there are some counter-examples to this assertion, given by Luo and Sarnak [87]; and they come from very strongly chaotic classical dynamics, so the source of the problem does not lie there.

The counter-examples are Laplacians on *arithmetic* hyperbolic surfaces (such as the modular surface and finite covers thereof); they are believed to be "non-generic" in some vague sense, and thus one may conjecture that the assertions above hold for "generic" systems. But even in such a weakened form, the question is fully open.

On the other hand, the question of localization of wave functions, although very difficult, has known steady progress in the last decade. In this paper, we will

- report on recent progress on delocalization of wave functions for chaotic systems, in the semiclassical limit (limit of small wavelengths), Section 2;
- discuss delocalization of eigenfunctions on large finite systems, such as large finite graphs, or Riemann surfaces of high genus, Sections 3 and 4;
- find a link between spatial delocalization and spectral delocalization on infinite systems, §3.4;
- note that delocalization of eigenvectors of large random matrices has also undergone intensive study lately. Although these eigenvectors have no direct physical

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interpretation, they are directly related to the Green function and were studied in relation with the question of universality of the spectrum. The spectacular recent progress on Wigner matrices and large random graphs will be mentioned in §4.1 – in a largely non exhaustive manner, as we will focus on results pertaining to delocalization of eigenvectors.

2. High frequency delocalization

In this section, we let (M, g) be a compact smooth Riemannian manifold of dimension d, and Δ be the Laplace-Beltrami operator on M. It is a self-adjoint operator on the Hilbert space $L^2(M, \mathrm{Vol})$, where Vol is the Riemannian volume measure. We diagonalize Δ : it is known that there is a non-decreasing sequence $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \longrightarrow +\infty$, and an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(M, \mathrm{Vol})$, such that

$$\Delta\phi_k = -\lambda_k\phi_k.$$

If M has a boundary, we impose a boundary condition, for instance the Dirichlet condition (i.e. we ask that ϕ_k vanishes on ∂M). The case when M is a billiard table, that is, a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary, already contains all the difficulties of the subject : actually, the presence of a boundary induces additional technical difficulties, and all the theorems given below have been proven for boundariless manifolds first.

In this part of the paper, we are interested in notions of delocalization defined in the high-frequency limit $\lambda_k \longrightarrow +\infty$. This is the same as the small wavelength limit, and it is also known as a *semiclassical limit*, meaning that classical dynamics emerges from quantum mechanics in this limit.

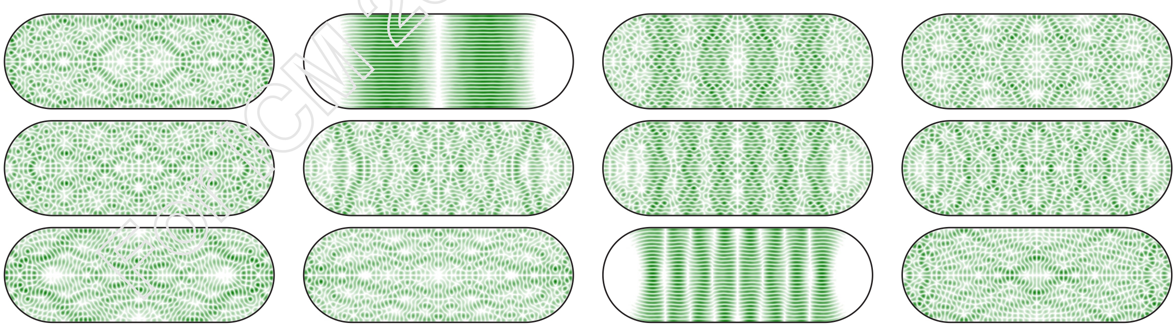

Figure 1. Plot of $|\phi_n(x, y)|^2$ for the stadium billiard with odd-odd symmetry, for consecutive states starting from $n = 319$. Darker shades correspond to large values of the eigenfunctions. Courtesy A. Bäcker

2.1. **The role of the geodesic flow .**

The eigenfunction equation $\Delta\phi_k = -\lambda_k\phi_k$ may be rewritten as $-\hbar^2\Delta\phi = E\phi$ (with $\lambda_k = E\hbar^{-2}$) to make a connection with the Schrödinger operators from (1) (so the external potential V vanishes here). If we impose that E stays away from 0, the limit $\lambda_k \longrightarrow +\infty$ is equivalent to $\hbar \longrightarrow 0$; in this régime, quantum mechanics should "converge to classical mechanics". This was actually a requirement of Schrödinger when he introduced his equation [100].

The Schrödinger operator $-\hbar^2\Delta$ corresponds to a particle moving on M in absence of any external force. In classical mechanics, this corresponds to the motion along geodesics, in other words, the motion with zero acceleration. When M is a billiard, the motion is in straight line, with reflection on the boundary. We denote by T^*M the *cotangent bundle* of M; this is the classical phase space. An element $(x,\xi) \in T^*M$ has a component $x \in M$ (the "position" of the particle) and $\xi \in T_x^*M$ (the "momentum"). For $(x,\xi) \in T^*M$, and $t \in \mathbb{R}$, we denote by $g^t(x,\xi) \in T^*M$ the position and momentum of the particle, after it has moved during time t along the geodesic starting at x with initial momentum ξ. The family $(g^t)_{t\in\mathbb{R}} : T^*M \longrightarrow T^*M$ is a flow of diffeomorphisms, meaning that $g^{t+s} = g^t \circ g^s$ and g^0 is the identity. This dynamical system is called the geodesic flow. The motion along geodesics has constant speed, and thus, the *unit cotangent bundle* $S^*M = \{(x,\xi) \in T^*M, \|\xi\|_x = 1\}$ is preserved by g^t.

In the limit of small wavelengths, ($\lambda_k \longrightarrow +\infty$), the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = i\Delta\psi$$

moves the *wavefronts* along geodesics. What we mean is that, if we start with an initial condition of the special form

$$\psi^\epsilon(x) = \chi(x)e^{i\frac{S(x)}{\epsilon}}$$

with χ and S smooth, and apply the unitary group $e^{i\epsilon t\Delta}$ (note the rescaling of time), then $e^{i\epsilon t\Delta}\psi^\epsilon$ is of the form

$$e^{i\epsilon t\Delta}\psi^\epsilon(x) = \chi_t(x)e^{i\frac{S(t,x)}{\epsilon}} + O_t(\epsilon) \tag{2}$$

where

- $S(t,x)$ satisfies the Hamilton-Jacobi equation $\frac{\partial S}{\partial t} = \|d_xS\|^2$; which means that at time t the "wavefront set" $\{(x, d_xS(t,x))\} \subset T^*M$ is the image under g^t of the initial wavefront set $\{(x, d_xS(x))\}$;
- denoting $G^t : x \mapsto \pi g^t(x, d_xS)$, where $\pi : T^*M \longrightarrow M$ is the projection to the position coordinate, we have
$$\chi_t(y) = \chi(G^{-t}y)|G^{-t}y|^{1/2}$$
where $|G^{-t}y|$ is the Jacobian of the map G^{-t} at y. This normalization factor is related to the fact that the L^2 norm must be constant in time.

Formula (2) is an approximate one, it has a remainder term $O_t(\epsilon)$. It is called a BKW approximation, as it was first obtained by Brillouin, Kramers, Wentzell [79, 115, 35]. Note that this description is usually only valid for small times; for larger times, it might no longer be possible to write the set $g^t\{(x, d_xS(x))\}$ in the form $\{(x, d_xS(t,x))\}$ for some smooth function $S(t,\cdot)$, as *caustics* may appear.

Using (2), and the fact that eigenfunctions satisfy $e^{i\epsilon t\Delta}\phi_k = e^{-i\epsilon t\lambda_k}\phi_k$, we can hope to establish a relation between the behaviour of the eigenfunctions and the large-time properties of the geodesic flow, however there are two main difficulties :

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– a general function ϕ is not of the form (2), but can be written as a linear superposition of such functions (with $\epsilon \sim \lambda_k^{-1/2}$ in the case of eigenfunctions). This may be seen using the Fourier transform in local coordinates. It is extremely difficult to control how the different terms will add up and interfere after applying $e^{i\epsilon t\Delta}$, for large t;
– the error term in (2) grows (usually exponentially) with t; so it is extremely delicate to use the approximation (2) for large times.

2.2. **L^p-norms as measures of delocalization ?** One of the first question that comes to mind at the sight of Figure 1 is : how large can the eigenfunctions be, how strongly can they be peaked, and at what points ? In this section we denote by ϕ_λ any solution of $-\Delta\phi_\lambda = \lambda\phi_\lambda$, normalized so that $\|\phi_\lambda\|_{L^2} = 1$. A general bound on the L^∞-norm is the following :

Theorem 1. *(known as Hörmander's bound)*

$$\|\phi_k\|_\infty = O(\lambda_k^{(d-1)/4}).$$

In a celebrated paper, C. Sogge gave a bound for all L^p norms, $2 \le p \le +\infty$:

Theorem 2 (Sogge, [107])**.**

$$\|\phi_\lambda\|_{L^p} = O(\lambda^{\frac{\mu(p)}{2}})$$

where

- $\mu(p) = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *for* $\frac{2(d+1)}{(d-1)} \le p \le +\infty$*;*
- $\mu(p) = \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$ *for* $2 \le p \le \frac{2(d+1)}{(d-1)}$*.*

These bounds hold for any compact manifold M. Recall that d is the dimension of M. Note the role of the critical value $p_c = \frac{2(d+1)}{(d-1)}$. The upper bounds are achieved on the sphere $\mathbb{S}^d$: with zonal spherical harmonics for $p \ge p_c$ (these spherical harmonics are strongly peaked at 2 poles), and with highest weight spherical harmonics for $p \le p_c$ (these spherical harmonics are peaked in the vicinity of a circle). So, in all the L^p-norms, the sphere is a case where eigenfunctions are most strongly peaked. For $p > p_c$, several results give a partial converse, showing that manifolds where the L^p-bound is saturated must have a "pole", that is, a point where many geodesics loops go through :

If $x \in M$, let $\mathcal{L}_x \subset S_x^*M$ be the set of directions that loop back to x, i.e.

$$\mathcal{L}_x = \{v \in S_x^*M, \exists t > 0, g^t(x,v) \in S_x^*M\}.$$

We denote by σ_x the Lebesgue measure on the sphere S_x^*M.

Theorem 3 (Sogge-Zelditch [108, 109, 110])**.**

- *Assume there exists a subsequence* $\lambda_{n_k} \longrightarrow +\infty$ *and* $C > 0$ *such that* $\|\phi_{\lambda_{n_k}}\|_\infty \ge C\lambda^{\frac{\mu(\infty)}{2}}$*. Then there exists* x *such that* $\sigma_x(\mathcal{L}_x) > 0$*;*
- *If* M *is real analytic, the existence of such subsequence* $\phi_{\lambda_{n_k}}$ *is* equivalent *to the existence of* x *such that* $\mathcal{L}_x = S_x^*M$*, and the first return map* $\eta_x : S_x^*M \longrightarrow S_x^*M$ *possesses an absolutely continuous invariant probability measure. (Moreover, in*

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that case, there exists $t_0 > 0$ such that $g^{t_0}(x, v) \in S^*_x M$ for all $v \in S^*_x M$, that is, there is a common return time).

- *If M is real analytic and $\dim M = 2$, the existence of such subsequence $\phi_{\lambda_{n_k}}$ is* equivalent *to the existence of $x \in M$ and $t_0 > 0$ such that $g^{t_0}(x, v) = (x, v)$ for all $v \in S^*_x M$.*

What about our original question ? is it true that, if the geodesic flow is chaotic, eigenfunctions will be much less peaked ? To be more specific, we shall mostly be interested in manifolds with negative sectional curvatures. It is then known that the geodesic flow has the *Anosov property*, which is a very strong and very well understood form of chaos : the geodesic flow is not only ergodic, it has strong mixing properties, is measurably isomorphic to a Bernoulli system, exhibits exponential sensitivity to initial conditions,... On a negatively curved manifold, there are only countably many closed geodesic, and what's more, through a point x there pass at most countably many geodesic loops. Thus, Theorem 3 implies that $\|\phi_\lambda\|_{L^\infty} = o\left(\lambda^{\frac{\mu(\infty)}{2}}\right)$ (the big O if Theorem 1 becomes a little o).

One can in fact go further :

Theorem 4. *(i)* (Bérard 1977 [17]) *If $d = 2$ and M has no conjugate points, or if $d \geq 2$ and M has non-positive sectional curvature, for $p = +\infty$,*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{\sqrt{\log \lambda}}\right).$$

(i') (Bonthonneau [30]) *Statement (i) actually holds if M has no conjugate points, for all $d \geq 2$.*

(ii) (Hassell-Tacy [63]) *(i) holds for all $p > p_c$.*

(iii) (Blair-Sogge [24, 22]) *If M has non-positive sectional curvature, for $p < p_c$, there exists $\sigma(p, d) > 0$ such that*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{(\log \lambda)^{\sigma(p,d)}}\right)$$

(iv) (Blair-Sogge [23]) *Statement (iii) still holds for $p = p_c$.*

(iii) and (iv) were previously proven by Hezari and Rivière [67] for negatively curved manifolds and for a density 1 sequence of eigenfunctions.

Although this logarithmic improvement constitutes a great progress, it is far from reaching our goal of saying that eigenfunctions are "spread around" if the geodesic flow is chaotic. In fact, after all it is only assumed that the curvature is non-positive, so the results hold already for flat tori (where the geodesic flow is completely integrable), and has not much to do with the long-term chaotic behaviour of the geodesic flow.

2.3. **The Shnirelman theorem and the Quantum Unique ergodicity conjecture .** As another indicator of delocalization, we can study the probability measure $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$. Ideally, the aim is to show that it is close to the uniform measure (say, asymptotically as $\lambda_k \longrightarrow +\infty$); or, maybe less ambitiously we could ask whether the measure $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$

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can be large on "small" sets (sets of small dimension for instance). The Quantum Ergodicity theorem gives a first and almost complete answer in case the geodesic flow is ergodic, with respect to the Liouville measure.

Recall, this means that for any L^1-function $a : S^*M \longrightarrow \mathbb{R}$, for Lebesgue almost-every $(x_0, \xi_0) \in S^*M$, the time average $\frac{1}{T}\int_0^T a \circ g^t(x_0, \xi_0)dt$ converges as $T \longrightarrow +\infty$ to the phase-space average $\int_{S^*M} a\, dL$ where L is the normalized Liouville measure on S^*M (i.e. the Lebesgue measure, the uniform measure), arising naturally from the symplectic structure on T^*M.

Quantum Ergodicity Theorem (Shnirelman theorem).

Theorem 5 (Shnirelman, Zelditch, Colin de Verdière [106, 40, 116])**.** *Let (M, g) be a compact Riemannian manifold, with the metric normalized so that* $\mathrm{Vol}(M) = 1$*. Call Δ the Laplace-Beltrami operator on M. Assume that the geodesic flow of M is* ergodic *with respect to the Liouville measure. Let $(\phi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M, g)$ made of eigenfunctions of the Laplacian*

$$\Delta\phi_k = -\lambda_k\phi_k, \qquad \lambda_k \leq \lambda_{k+1} \longrightarrow +\infty.$$

Let a be a continuous function on M. Then

$$\frac{1}{N(\lambda)} \sum_{k, \lambda_k \leq \lambda} \left| \langle \phi_k, a\phi_k \rangle_{L^2(M)} - \int_M a(x) d\,\mathrm{Vol}(x) \right|^2 \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0 \tag{3}$$

where the normalizing factor is $N(\lambda) = |\{k, \lambda_k \leq \lambda\}|$.

Note that $\langle \phi_k, a\phi_k \rangle_{L^2(M)} = \int_M a(x)|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$.

Remark 6. *The Cesaro limit* (3) *implies that there exists a subset $S \subset \mathbb{N}$ of density* 1 *such that*

$$\langle \phi_k, a\phi_k \rangle \underset{n \longrightarrow +\infty, n \in S}{\longrightarrow} \int_M a(x) d\,\mathrm{Vol}(x). \tag{4}$$

In addition, using the fact that the space of continuous functions is separable, one can actually find $S \subset \mathbb{N}$ of density 1 *such that* (4) *holds for* all $a \in C^0(M)$*. In other words, the sequence of measures $(|\phi_k(x)|^2 d\,\mathrm{Vol}(x))_{n\in S}$ converges weakly to the uniform measure $d\,\mathrm{Vol}(x)$.*

Actually, the full statement of the theorem says that there exists a subset $S \subset \mathbb{N}$ of density 1 *such that*

$$\langle \phi_k, A\phi_k \rangle \underset{n \longrightarrow +\infty, n \in S}{\longrightarrow} \int_{S^*M} \sigma^0(A) dL \tag{5}$$

for every pseudodifferential operator A of order 0 *on M. On the right-hand side, $\sigma^0(A)$ is the principal symbol of A, that is a function on the unit cotangent bundle S^*M. Equation 4 corresponds to the case where A is the operator of multiplication by the function a.*

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The theorem has subsequently been extended to manifolds with boundary [57, 119]. It applies, in particular, to the stadium billiard in Figure 1, where the billiard flow has been proven by Bunimovich to be ergodic. The observation of large samples of eigenfunctions reveals that, indeed, most eigenfunctions are uniformly distributed over the stadium, but some of them look very localized inside the rectangle, and some of them also exhibit some mild enhancement in the neighbourhood of unstable periodic orbits, a phenomenon called "scarring" by physicists (Heller,[66]).

The theorem was also extended to general Schrödinger operators (or even pseudodifferential operators) in the limit $\hbar \longrightarrow 0$ [65]; more recently, to systems of differential operators acting on sections of vector bundles – such as Dirac operators, Dolbeault Laplacians,... [29, 71, 72]. The case of metrics with jump-like discontinuities has been elucidated [70], as well as the case of pseudo-riemannian Laplacians on 3-dimensional contact manifolds (for instance, the Laplacian on the Heisenberg group or its quotients) [42]. "Small scale quantum ergodicity", that is, the possibility to use in (3) a test function a whose support shrinks as $\lambda_k \longrightarrow +\infty$, has been explored in [67, 60] on negatively curved manifolds, and on flat tori in [61, 83].

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Quantum Unique Ergodicity conjecture. One may wonder whether the full sequence converges in (5), without having to extract the subsequence S. Figure 1 (or larger samples of eigenfunctions) suggests that this is not the case for the billiard stadium, where we see a sparse sequence of eigenfunctions that are not at all equidistributed.

This was proven by Hassell in 2008 [62] (for "almost all" stadium billiards, meaning, for Lebesgue-almost-all lengths of the stadium).

On the other hand, Rudnick and Sarnak's Quantum Unique Ergodicity (QUE) conjecture [95] predicts that if M is a compact boundaryless manifold with *negative sectional curvatures*, then one has convergence of the full sequence in (5), in other words the whole sequence of eigenfunctions becomes equidistributed as $\lambda \longrightarrow +\infty$. The conjecture has been proved by Lindenstrauss in the setting of "Arithmetic Quantum Unique Ergodicity", where M is an "arithmetic" hyperbolic surface, and where the ϕ_λ are assumed to be eigenfunctions, not only of the Laplacian, but also of the Hecke operators [84, 34, 39].

Arithmetic Quantum Unique Ergodicity will not be discussed with enough detail in this text, but the results have been presented at previous ICM's; we refer to [98, 48, 85, 111] for a more adequate overview.

For general negatively curved manifolds, the conjecture is open, but in the last 20 years significant progress has been made :

2.4. **Entropy and support of semiclassical measures.** In this section, M is assumed to have negative sectional curvature, and dimension d.

Let us come back to the diagonal matrix elements $\langle \phi_n, A\phi_n \rangle$ appearing in (5), where A is a pseudodifferential operator of order 0. By a general compactness argument, one may always extract subsequences so that $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converge for all A. The limit is of the form $\int_{S^*M} \sigma^0(A) d\mu$, where μ is a probability measure on S^*M. A measure obtained this way is obtained, according to sources, "microlocal defect measure", "semiclassical measure", or "microlocal lift" associated with the sequence ϕ_{n_k}. The Quantum Unique Ergodicity conjecture described above is equivalent to proving that μ has to be the Liouville measure, for every subsequence (ϕ_{n_k}). But without aiming that far, we can try to characterize specific properties of the measure μ. A priori, we only know that μ has to be invariant under the geodesic flow : that is, $g^t_\sharp \mu = \mu$ for all $t \in \mathbb{R}$. This is a consequence of the eigenfunction property and of the classical/quantum correspondence as $\lambda \longrightarrow +\infty$, as seen in §2.1.

Theorem 7. [3] *Assume M is a compact Riemannian manifold with negative sectional curvature. Assume $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converges to $\int_{S^*M} \sigma^0(A) d\mu$ for all A. Then μ has positive entropy.*

This is the Kolmogorov-Sinai entropy of dynamical systems. We do not give its definition here, but state a few facts to help understand the implications of the theorem. To each invariant probability measure ν of a dynamical system (here the geodesic flow), one can associate a non-negative number $h_{KS}(\nu)$, having the following properties :

- if ν is carried by a periodic trajectory, then $h_{KS}(\nu) = 0$;
- $\nu \mapsto h_{KS}(\nu)$ is affine : $h_{KS}(\alpha\nu_1 + (1-\alpha)\nu_2) = \alpha h_{KS}(\nu_1) + (1-\alpha) h_{KS}(\nu_2)$, for any invariant measures ν_1, ν_2, for $\alpha \in [0, 1]$;
- (Pesin-Margulis-Ruelle inequality) if the dynamical system is sufficiently smooth,

$$h_{KS}(\nu) \leq \int \left(\sum_{j=1}^{r} \chi_j^+(\rho) \right) d\nu(\rho) \tag{6}$$

  where the numbers $\chi_j^+(\rho)$ are the positive Lyapunov exponents of a point ρ – defined ν-almost everywhere, by the Oseledets theorem – that give the rate of exponential instability of the trajectory of ρ.
- in the case of the geodesic flow on a negatively curved manifold, there is equality in (6) if and only if ν is the Liouville measure L (Ledrappier-Young [81, 82]);
- in the case of the geodesic flow on a negatively curved manifold, of constant curvature -1 and dimension d (so that S^*M has dimension $2d-1$), there are $d-1$ positive Lyapunov exponents, they do not depend on ρ and have the value 1. Thus, (6) can be written as

$$h_{KS}(\nu) \leq d - 1, \tag{7}$$

  with equality if and only if ν is the Liouville measure L.

Let us give two more transparent corollaries to Theorem 7 :

Corollary 1. *Let $\Gamma \subset S^*M$ be the union of all points lying on a periodic trajectory on the geodesic flow (recall, if M has negative curvature, there are countably many periodic geodesics). Let μ be as in Theorem 7. Then $\mu(\Gamma) < 1$.*

Otherwise, μ would have zero entropy. In the physics literature, an eigenfunction that is enhanced near an unstable periodic classical trajectory is said to have a *scar* (Heller, [66]). In the mathematics literature, a sequence of eigenfunctions is said to be *strongly scarred*

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if the corresponding semiclassical meaure μ is supported on some periodic trajectory. Our theorem thus shows that this is not possible on a negatively curved manifold (however, it does not rule out a *partial scar*, that is to say that $\mu(\Gamma) > 0$).

From the definition of entropy, one can also prove :

Corollary 2. *The support of μ has Hausdorff dimension > 1.*

Note that the fact that the dimension is ≥ 1 is trivial since μ is invariant under the geodesic flow.

With Nonnenmacher, we later obtained a more quantitative version if the curvature is constant.

Theorem 8 (Anantharaman-Nonnenmacher [7])**.** *Assume M is a compact Riemannian manifold of dimension d, with constant sectional curvature -1. Then μ has entropy greater than $\frac{d-1}{2}$.*

By the aforementioned properties of entropy, the QUE conjecture in constant negative curvature is equivalent to proving that μ has entropy $d-1$, so we fall short of a factor $1/2$. There are toy models of quantum chaos where it is known that the lower bound $\frac{d-1}{2}$ is sharp, i.e. there are sequences of eigenfunctions that are not equistributed and have exactly half the maximal entropy : see the quantum cat map and the quantum baker's map [55, 6].

Corollary 3. *The support of μ has Hausdorff dimension $\geq d$.*

As a comparison, the dimension of the full phase space T^*M is $2d$, and of the energy layer S^*M is $2d-1$.

Corollary 4. *Let $\Gamma \subset S^*M$ be the union of all points lying on a closed trajectory on the geodesic flow. Let μ be as in Theorem 7, with M of constant negative curvature. Then $\mu(\Gamma) \leq 1/2$.*

Indeed, let us decompost μ as $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, where μ_1 is carried by Γ, so that $h_{KS}(\mu_1) = 0$. So $h_{KS}(\mu) = (1-\alpha)h_{KS}(\mu_2)$. The result says that this has to be $\geq \frac{d-1}{2}$; but the entropy of μ_2 is smaller than the maximal entropy $d-1$, so necessarily $\alpha \leq 1/2$. For the toy model of the quantum cat map, Corollary 4 had been proven by Faure and Nonnenmacher in [54] without using entropy.

In variable curvature, the generalization of Theorem 8 should be that the entropy of μ is greater than $\frac{1}{2}\int_{S^*M}\sum_{j=1}^{d-1}\chi_j^+ d\mu$, where χ_j^+ are the Lyapunov exponents. However our method in [7] gives a slightly less good bound; the predicted lower bound in variable curvature has only been obtained for $d=2$, by G. Rivière [94]. Again, by the Ledrappier-Young theorem [81], proving QUE is equivalent to getting rid of the factor $1/2$ in Rivière's result.

Theorem 9 (Dyatlov-Jin [47])**.** *μ has full support, that is, $\mu(\Omega) > 0$ for any non-empty open set $\Omega \subset S^*M$.*

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Note that Theorems 7 and 9 are somehow independent. There are measures with positive entropy and not full support (for instance, measures supported by geodesics avoiding an open set Ω may have a large entropy). And there are measures having full support but zero entropy (for instance, a measure putting positive weight on each periodic geodesic). Both results leave open the question whether μ can be a convex combination of the Liouville measure and a measure carried on a closed geodesic. Such limit measures appeared in the aforementioned toy models of quantum chaos [55].

2.5. **Some questions on non-compact manifolds.** We have chosen to limit the scope of this text to compact manifolds (and thus, "delocalization" is understood in the limit of small wavelength, but does not deal with what happens at infinity). There are of course many interesting questions related to delocalization phenomena on non-compact manifolds, that we briefly review in this paragraph.

In keeping with the rest of this paper, let us consider the Laplacian on a non-compact riemannian manifold M (most questions also make sense for general Schrödinger operators).

2.5.1. *Absolutely continuous spectrum* . In the context of infinite systems, the word "delocalization" is often used to mean that the Laplacian has no pure-point spectrum (this means that eigenfunctions are not square-integrable), or even stronger, purely absolutely continuous spectrum, in some region of the spectrum.

As we will see in §4.3, one can sometimes prove that this implies a form of "quantum ergodicity" for eigenfunctions on large compact manifolds approximating M (see Theorem 22 for a precise statement). For the moment, this theorem is restricted to the case where M is the hyperbolic disc, where the spectrum of the Laplacian is explicit and can be seen (by direct computation) to be purely absolutely continuous. In general, it turns out to be very difficult to find examples of M having purely absolutely continuous in some interval of the L^2-spectrum, outside of the world of locally symmetric spaces. For instance, starting from X a compact riemannian manifold with *variable* negative sectional curvature, and taking $M = \widetilde{X}$ to be its universal cover, it seems that nothing is known about the nature of the spectrum of the Laplacian on M, although one would be naturally inclined to guess that is has absolutely continuous spectrum.

2.5.2. *Large frequency delocalization on non-compact manifolds.* If ϕ_λ is a solution of $-\Delta\phi_\lambda = \lambda\phi_\lambda$ on M (with $\lambda \in \mathbb{R}$) one could ask the question of the behaviour of the measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$, in the limit $\lambda \longrightarrow +\infty$, even if M is non-compact. More precisely, it seems reasonable to restrict these measures to a compact set before studying this limit.

When M is a *finite volume* hyperbolic surface – so that the ends of M are hyperbolic cusps – the question was studied by Zelditch in [117], for ϕ_λ generalized eigenfunctions corresponding to the absolutely continuous spectrum : the so-called Eisenstein series. A "quantum ergodicity" theorem was proven. It was strengthened to a "quantum unique ergodicity" result by Jakobson in [69], when M is the modular surface. When M has variable curvature in a compact subset, but still has hyperbolic cusps, quantum ergodicity was proven more recently in [31]. Note that for *infinite volume*, convex-cocompact manifolds,

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quantum (unique) ergodicity for the Eisenstein series has been studied in [46, 59, 68] but the phenomena are quite different, as it is not the Liouville measure that appears at the semiclassical limit, but a family of measures indexed by the boundary at infinity.

Back to the case of finite volume hyperbolic surface, an example of special interest in number theory is the modular surface and its congruence covers. In this case, M has an infinite sequence of discrete eigenvalues embedded in the continuous spectrum (see the survey papers [96, 98] for more details and references). "Arithmetic quantum ergodicity" is the study of the joint L^2-eigenfunctions of the Laplacian and of the so-called "Hecke operators". In this context, Arithmetic Quantum Unique ergodicity, that is, the convergence of the full sequence of probability measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$ to a multiple of the uniform measure, was proven by Lindenstrauss [84]. Since the modular surface is not compact, there can be escape of mass to infinity, and thus it is not clear that the limit of the measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$ is still a probability measure. Escape of mass was ruled out by Soundararajan [112].

Having discrete spectrum embedded in the continuous spectrum is non-generic. For general hyperbolic surfaces, the discrete spectrum is turned into the "resonance spectrum"; resonances are poles of the analytic continuation on the resolvent restricted to $C_c^\infty(M)$ [102, 32]. Generically, resonances are not real. Naturally attached to resonances, there are non-L^2-eigenstates called "resonant states". The question of quantum ergodicity for resonant states is to this date fully open, and seems extremely difficult.

3. Large scale delocalization

In the mathematical physics literature, it is believed that the spectrum of the Laplacian, as well as its eigenfunctions, should exhibit universal features that depend only on qualitative geometric properties of the space. Localization/delocalization of eigenfunctions is believed to bear close relation with the nature of *spectral statistics* : localization is supposedly associated with Poissonian spectral statistics, whereas delocalization should be associated with Random Matrix statistics (GOE/GUE). In the field of quantum chaos, the former notion is often associated with *integrable dynamics* and the latter with *chaotic dynamics* [21, 26]. However, specific examples show that the relation is not so straightforward [87, 96, 97, 88]. Understanding how far one can push these ideas is one amongst many reasons for studying models of large graphs as toy models [73, 77, 78, 104, 105].

It seems that "quantum graphs" have been studied before discrete graphs in the context of quantum chaos. By "quantum graphs", we mean 1-dimensional CW-complexes with $\Delta = \frac{d^2}{dx^2}$ on the edges and suitable matching conditions on the vertices; the most natural ones being the "Kirchhoff" matching condition where it is asked that the functions are continuous at the vertices, and that the sum of their derivatives at a vertex vanish. On a fixed quantum graph, it is known that the analogue of Shnirelman's theorem never holds in the large frequency limit $\lambda \longrightarrow +\infty$ [41]. See also [19, 74, 58, 18] for other results pertaining to eigenvalue or eigenfunction statistics on compact quantum graphs.

In what follows, instead of the high-frequency limit, we consider the limit where the size of the graph goes to infinity ("large scale limit"). We focus on discrete graphs and the

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eigenfunctions of their adjacency operators – although similar questions for large quantum graphs should also be explored in the future. We mostly focus on discrete *regular* graphs, but in §3.4 also report on recent progress concerning non-regular graphs.

3.1. **Overview of the problem.** Consider a very large graph $G = (V, E)$. Are the eigenfunctions of its adjacency matrix *localized*, or *delocalized* ? These words are used in a variety of contexts, with several different meanings.

For discrete Schrödinger operators on infinite graphs (e.g. for the celebrated Anderson model describing the metal-insulator transition), localization can be understood in a spectral, spatial or dynamical sense. Given an interval $I \subset \mathbb{R}$, one can consider

- *spectral localization :* pure point spectrum in I,
- *exponential localization :* the corresponding eigenfunctions decay exponentially,
- *dynamical localization :* an initial state with energy in I which is localized in a bounded domain essentially stays in this domain as time goes on.

On the opposite, delocalization may be understood at different levels :

- *spectral delocalization :* purely absolutely continuous spectrum in I,
- *ballistic transport :* wave packets with energies in I spread on the lattice at a specific (ideally, linear) rate as time goes on.

Here we want to discuss notions of *spatial delocalization*. Since the wavefunctions corresponding to absolutely continuous spectrum are not square-summable, a natural interpretation of spatial delocalization is to consider a sequence of growing "boxes" or finite graphs (G_N) approximating the infinite system in some sense, and ask if the eigenfunctions on (G_N) become delocalized as $N \longrightarrow \infty$. Can they concentrate on small regions, or, on the opposite, are they uniformly distributed over (G_N) ? Large, finite graphs are also a subject of interest on their own. Actually, an infinite system is often an idealized version of a large finite one.

Recently, the question of delocalization of eigenfunctions of large matrices or large graphs has been a subject of intense activity. Let us mention several ways of testing delocalization that have been used. Let M_N be a large symmetric matrix of size $N \times N$, and let $(\phi_j)_{j=1}^N$ be an orthonormal basis of eigenfunctions. The eigenfunction ϕ_j defines a probability measure $\sum_{x=1}^N |\phi_j(x)|^2 \delta_x$. The goal is to compare this probability measure with the uniform measure, which puts mass $1/N$ on each point.

- *ℓ^∞ norms :* Can we have a pointwise upper bound on $|\phi_j(x)|$, in other words, is $\|\phi_j\|_\infty$ small, and how small compared with $1/\sqrt{N}$?
- *ℓ^p norms:* Can we compare $\|\phi_j\|_p$ with $N^{1/p-1/2}$? In [1], a state ϕ_j is called *non-ergodic* (and *multi-fractal*) if $\|\phi_j\|_p$ behaves like $N^{f(p)}$ with $f(p) \neq 1/p - 1/2$.
- *Scarring :* Can we have full concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq 1 - \epsilon$) or partial concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq \epsilon$) with Λ a set of "small" cardinality ? We borrow the term "scarring" from the term used in the theory of quantum chaos [66].
- *Quantum ergodicity :* Given a function $a : \{1, \ldots, N\} \longrightarrow \mathbb{C}$, can we compare $\sum_x a(x)|\phi_j(x)|^2$ with $\frac{1}{N}\sum_x a(x)$? This criterion is borrowed again from quantum chaos, it is inspired from the Shnirelman theorem 5. It was applied to discrete

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regular graphs in [5, 4]. Quantum ergodicity means that the two averages are close for *most* j. If they are close for *all* j, one speaks of *quantum unique ergodicity*.

As was demonstrated in a recent series of papers by Yau, Erdös, Schlein, Knowles, Bourgade, Bauerschmidt, Yin, Huang... adding some randomness may allow to settle the problem completely, proving *almost sure* optimal ℓ^∞-bounds and quantum unique ergodicity for various models of *random* matrices and *random* graphs, such as Wigner matrices, sparse Erdös-Rényi graphs, random regular graphs of slowly increasing or bounded degrees [52, 53, 33, 51, 15, 13, 14] : see §4.1. The completely different point of view adopted in [38, 5] is to consider deterministic graphs and to prove delocalization as resulting directly from the geometry of the graphs.

3.2. **Entropy.** The paper [38] by Brooks and Lindenstrauss has pioneered the study of the spatial distribution of eigenfunctions of the Laplacian on large deterministic $(q+1)$-regular graphs (that is, such that each vertex has the same number of neighbours, denoted by $q+1$).

Consider a sequence of $(q+1)$-regular connected graphs $(G_N)_{N\in\mathbb{N}} = (V_N, E_N)$. Consider the adjacency operator defined on functions on V_N by

$$\mathcal{A}_N f(x) = \sum_{x\sim y} f(y) \tag{8}$$

where $x \sim y$ means x and y are related by an edge. The discrete Laplacian is

$$\Delta_N f(x) = \sum_{x\sim y} \left(f(y) - f(x)\right). \tag{9}$$

For regular graphs these two operators are essentially the same :

$$\mathcal{A}_N - (q+1)I = \Delta_N. \tag{10}$$

Theorem 10 (Brooks-Lindenstrauss [38])**.** *Let (G_N) be a sequence of $(q+1)$-regular graphs (with q fixed), $G_N = (V_N, E_N)$ with $V_N = \{1, \dots, N\}$. Assume that*[1] *there exists $c > 0, \delta > 0$ such that, for any $k \leq c \ln N$, for any pair of vertices $x, y \in V_N$,*

$$|\{\text{paths of length } k \text{ in } G_N \text{ from } x \text{ to } y\}| \leq q^{k\left(\frac{1-\delta}{2}\right)}. \tag{11}$$

Fix $\epsilon > 0$. Then, if ϕ is an eigenfunction of the discrete Laplacian on G_N and if $\Lambda \subset V_N$ is a set such that

$$\sum_{x\in\Lambda} |\phi(x)|^2 \geq \epsilon \sum_{x\in V_N} |\phi(x)|^2,$$

then $|\Lambda| \geq N^\alpha$ — where $\alpha > 0$ is given as an explicit function of ϵ, δ and c.

This theorem is reminiscent of Theorems 7 and 8 about the entropy of eigenfunctions in the large frequency limit. It is stronger than saying that the entropy

$$H_N(\phi) = -\frac{1}{\log N} \sum_x |\phi(x)|^2 \ln |\phi(x)|^2$$

[1] This assumption holds in particular if the injectivity radius is $\geq c \ln N$. The interest of the weaker assumption is that it holds for typical random regular graphs [90].

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is bounded from below by a positive constant.

A careful reading also reveals that the proof shows some logarithmic upper bound on the L^∞-norm of eigenfunctions : $\|\phi\|_\infty = O((\log N)^{-1/4})$. Very recently, Brooks and Le Masson have announced an improvement of the power 1/4 under a stronger assumption than (11) [36].

3.3. **QE on regular graphs.** In [5], a general statement of "quantum ergodicity" was obtained for the first time in the large scale limit, namely for the discrete Laplacian on large regular graphs. We consider a sequence $G_N = (V_N, E_N)$ of $(q+1)$-regular graphs, and now assume the following :

(EXP) The sequence of graphs is a family of expanders. More precisely, there exists $\beta > 0$ such that the spectrum of $(q+1)^{-1}\mathcal{A}_N$ on $\ell^2(V_N)$ is contained in $\{1\}\cup[-1+\beta, 1-\beta]$ for all N.

Note that 1 is always an eigenvalue, corresponding to constant functions. Our assumption implies in particular that each G_N is connected and non-bipartite. It is well-known that a uniform spectral gap for $\mathcal{A}_N$ is equivalent to a Cheeger constant bounded away from 0, which means that the graph is very connected (see for instance [44], §3).

(BST) For all R,

$$\frac{|\{x \in V_N, \rho(x) < R\}|}{N} \underset{N\longrightarrow\infty}{\longrightarrow} 0$$

where $\rho(x)$ is the "injectivity radius" of x, that is to say, the largest integer r such that the ball $B(x, r)$ is a tree.

(BST) can be rephrased by saying that our sequence of graphs converges, in the sense of Benjamini-Schramm [16], to the $(q+1)$-regular tree. In particular, this condition is satisfied if the girth goes to infinity. In what follows we denote by $\mathfrak{X}$ the $(q+1)$-regular tree. Condition **(BST)** implies the convergence of the spectral measure, according to the Kesten-McKay law [75, 89]. Call $(\lambda_1^{(N)}, \ldots, \lambda_N^{(N)})$ the eigenvalues of $\mathcal{A}_N$ on G_N; then, for any interval $I \subset \mathbb{R}$,

$$\frac{1}{N}|\{j, \lambda_j^{(N)} \in I\}| \underset{N\longrightarrow+\infty}{\longrightarrow} \int_I \mathrm{m}(\lambda)d\lambda$$

where $\mathrm{m}(\lambda)$ is a probability density corresponding to the spectral measure of a Dirac mass δ_o for the operator $\mathcal{A}$ on $\ell^2(\mathfrak{X})$. This measure can be characterized by its moments,

$$\int \lambda^k \mathrm{m}(\lambda)d\lambda = \langle \delta_o, \mathcal{A}_{\mathfrak{X}}^k \delta_o\rangle_{\ell^2(\mathfrak{X})} \tag{12}$$

where $\mathcal{A}_{\mathfrak{X}}$ is the adjacency operator on $\mathfrak{X}$; this is also the number of paths $\mathfrak{X}$, starting at o and returning to o after k steps. We won't need the explicit expression of m here, but let us mention that it is smooth and positive on $(-2\sqrt{q}, 2\sqrt{q})$ and vanishes elsewhere. This implies that most of the eigenvalues $\lambda_j^{(N)}$ are in $(-2\sqrt{q}, 2\sqrt{q})$, an interval strictly smaller than $[-(q+1), q+1]$.

The main result of [5, 4] is stated below as Theorem 11.

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Theorem 11 ([5] Anantharaman-Le Masson)**.** *Let $(G_N) = (V_N, E_N)$ be a sequence of $(q+1)$-regular graphs with $|V_N| = N$. Assume that (G_N) satisfies* **(BST)** *and* **(EXP)***.*

Let $(\phi_1^{(N)}, \ldots, \phi_N^{(N)})$ be an orthonormal basis of eigenfunctions of $\mathcal{A}_N$ in $\ell^2(V_N)$.

Let $a_N : V_N \longrightarrow \mathbb{C}$ be a sequence of functions such that $\sup_N \sup_{x\in V_N} |a_N(x)| \leq 1$. Define $\langle a_N \rangle = \frac{1}{N}\sum_{x\in V_N} a_N(x)$.

Then

$$\frac{1}{N}\sum_{j=1}^{N}\Big|\langle \phi_j^{(N)}, a_N\phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\Big|^2 \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

Equivalently, for any $\delta > 0$,

$$\frac{1}{N}\Big|\Big\{j \in [1, N], \Big|\langle \phi_j^{(N)}, a_N\phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\Big| > \delta\Big\}\Big| \underset{N\longrightarrow+\infty}{\longrightarrow} 0. \tag{13}$$

Note that $\langle \phi_j^{(N)}, a_N\phi_j^{(N)}\rangle_{\ell^2(V_N)}$ is the scalar product between $\phi_j^{(N)}$ and $a_N\phi_j^{(N)}$, its explicit expression is $\sum_{x\in V_N} a_N(x)|\phi_j^{(N)}(x)|^2$. The interpretation of Theorem 11 is that we are trying to measure the distance between the two probability measures on V_N,

$$\sum_{x\in V_N} |\phi_j^{(N)}(x)|^2\delta_x \quad \text{and} \quad \frac{1}{N}\sum_{x\in V_N}\delta_x \quad \text{(uniform measure)}$$

in a rather weak sense (just by testing the function a_N against both). What (13) tells us is that for large N and for most indices j, this distance is small.

3.4. **Non-regular graphs : from spectral to spatial delocalization.** The results described up to now only deal with *regular* graphs. The proofs always use, in some way or the other, the explicit Fourier analysis infinite regular trees. The aim of the paper [9] was to extend the *quantum ergodicity theorem* to eigenfunctions of discrete Schrödinger operators on quite general large graphs. A particularly interesting point of the result below is that it gives a direct relation between *spectral delocalization* of infinite systems and *spatial delocalization* of large finite system. The result may be summarized as follows (with proper additional assumptions to be described later) :

"If a large finite system is close (in the Benjamini-Schramm topology) to an infinite system having purely absolutely continuous spectrum in an interval I, then the eigenfunctions (with eigenvalues lying in I) of the finite system satisfy quantum ergodicity."

We consider a sequence of connected graphs without self-loops and multiple edges $(G_N)_{N\in\mathbb{N}}$. We assume each vertex has at least 3 and at most d neighbours.

We denote by V_N and E_N the vertices and edges of G_N, respectively. We assume $|V_N| = N$ and work in the limit $N \longrightarrow \infty$. Define the adjacency operator $\mathcal{A}_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$ by

$$(\mathcal{A}_N f)(v) = \sum_{w\sim v} f(w)\,,$$

where $v \sim w$ means v and w are nearest neighbours. The central object of our study are the eigenfunctions of $\mathcal{A}_N$, and their behaviour (localized/delocalized) as $N \longrightarrow +\infty$.

We shall assume the following conditions on our sequence of graphs:

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(EXP) The sequence (G_N) forms an expander family.

More precisely, for a non-regular graph, let us define the Laplacian (generator of the simple random walk) $P_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$ by

$$(P_N f)(x) = \frac{1}{d_N(x)} \sum_{y \sim x} f(y)\,, \tag{14}$$

where $d_N(x)$ stands for the number of neighbours of x. **(EXP)** means that the Laplacian P_N has a uniform spectral gap, that is, the eigenvalue 1 of P_N is simple, and the spectrum of P_N is contained in $[-1+\beta, 1-\beta] \cup \{1\}$, where $\beta > 0$ is independent of N.

Note that 1 is always an eigenvalue, corresponding to constant functions. Our assumption implies in particular that each G_N is connected and non-bipartite. It is well-known that a uniform spectral gap for P_N is equivalent to a Cheeger constant bounded away from 0 (see for instance [44], §3).

Our second assumption is that (G_N) has few short loops:

(BST) For all $r > 0$,

$$\lim_{N\to\infty} \frac{|\{x \in V_N : \rho_{G_N}(x) < r\}|}{N} = 0\,,$$

where $\rho_{G_N}(x)$ is the *injectivity radius* at x, i.e. the largest ρ such that the ball $B_{G_N}(x, \rho)$ is a tree.

The general theory of Benjamini-Schramm convergence (or local weak convergence [16]), allows us to assign a limit object to the sequence (G_N), which is a probability distribution on the set of *rooted graphs* (modulo isomorphism). More precisely, up to passing to a subsequence, assumption **(BST)** above is equivalent to the following assumption.

(BSCT) (G_N) converges in the local weak sense to a random of rooted *tree* $[\mathcal{T}, o]$.

Let us denote $\mathbb{P}$ the law of $\{[\mathcal{T}, o]\}$; thus $\mathbb{P}$ is a probability measure on the space of rooted trees.

Call $(\lambda_j^{(N)})_{j=1}^N$ the eigenvalues of $\mathcal{A}_N$ on $\ell^2(V_N)$. Assumption **(BSCT)** implies the convergence of the empirical law of eigenvalues : for any continuous $\chi : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\frac{1}{N}\sum_{j=1}^{N} \chi(\lambda_j^{(N)}) \underset{N\longrightarrow+\infty}{\longrightarrow} \mathbb{E}\left(\langle \delta_o, \chi(\mathcal{A}_\mathcal{T})\delta_o\rangle\right) =: \int \chi(t) d\mathrm{m}(t)\,, \tag{15}$$

where $\mathcal{A}_\mathcal{T}$ is the adjacency matrix of $\mathcal{T}$, it is a self-adjoint operator on $\ell^2(\mathcal{T})$. Here $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. The measure m is called the *integrated density of states* in the theory of random Schrödinger operators.

The forthcoming assumption is rather technical to state; it says - in a strengthened manner - that there is an interval I in which the spectrum of $\mathcal{A}_\mathcal{T}$ is absolutely continuous (for $\mathbb{P}$-almost every $[\mathcal{T}, o]$). Let $[\mathcal{T}, o]$ be a rooted tree (chosen randomly according to the law $\mathbb{P}$). Given $x, y \in \mathcal{T}$, and $\gamma \in \mathbb{C} \setminus \mathbb{R}$, we introduce the Green function

$$\mathcal{G}^\gamma(x, y) = \langle \delta_x, (\mathcal{A}_\mathcal{T} - \gamma)^{-1}\delta_y\rangle_{\ell^2(\mathcal{T})}\,.$$

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Given $v, w \in \mathcal{T}$ with $v \sim w$, we denote by $\mathcal{T}^{(v|w)}$ the tree obtained by removing from the tree $\mathcal{T}$ the branch emanating from v that passes through w. We denote by $\mathcal{A}_{\mathcal{T}^{(v|w)}}$ the corresponding adjacency matrix, and by $\mathcal{G}^{(v|w)}(\cdot,\cdot;\gamma)$ the corresponding Green function. We then put $\zeta_w^\gamma(v) := -\mathcal{G}^{(v|w)}(v,v;\gamma)$.

(Green) There is a non-empty open set I, such that for all $s > 0$ we have

$$\sup_{\lambda\in I,\eta_0\in(0,1)} \mathbb{E}\left(\sum_{y:y\sim o} |\operatorname{Im}\zeta_o^{\lambda+i\eta_0}(y)|^{-s}\right) < \infty\,.$$

To understand the implications of **(Green)**, define the (rooted) spectral measure of $[\mathcal{T}, o]$ by

$$\mu_o(J) = \langle \delta_o, \mathbb{1}_J(\mathcal{A}_{\mathcal{T}})\delta_o\rangle \qquad \text{for Borel } J \subseteq \mathbb{R}\,. \tag{16}$$

It can be shown that Assumption **(Green)** implies that $\sup_{\lambda\in I,\eta_0>0}\mathbb{E}(|\mathcal{G}^\gamma(o,o)|^2) < \infty$. As shown by Klein in [76], this implies that for $\mathbb{P}$-a.e. $[\mathcal{T}, o]$, the spectral measure μ_o is absolutely continuous in I, with density $\frac{1}{\pi}\operatorname{Im}\mathcal{G}^{\lambda+i0}(o,o)$. Hence, **(Green)** implies that $\mathbb{P}$-a.e. operator $\mathcal{A}_{\mathcal{T}}$ has purely absolutely continuous spectrum in I. This is a natural assumption since we wish to interprete Theorem 12 as a delocalization property of eigenfunctions. Negative moments such as **(Green)**, with $s < 0$, were used in the work by Aizenman and Warzel [2] to show ballistic transport for the Anderson model on the regular tree, that is, a form of delocalization for the time-dependent Schrödinger equation.

Let us state the main abstract result.

Let I be the open set of Assumption **(Green)**, and let us fix an interval I_1 (or finite union of intervals) such that $\bar{I}_1 \subset I$. We write G_N as a quotient $\Gamma_N\backslash\widetilde{G_N}$ where $\widetilde{G_N}$ is a tree (the universal cover of G_N). For $\tilde{x}, \tilde{y}$ vertices of $\widetilde{G_N}$, and $\gamma \in \mathbb{C}\setminus\mathbb{R}$, we introduce the Green function of the adjacency matrix $\widetilde{\mathcal{A}}_N$ of $\widetilde{G_N}$

$$\tilde{g}_N^\gamma(\tilde{x},\tilde{y}) = \langle \delta_{\tilde{x}}, (\widetilde{\mathcal{A}}_N - \gamma)^{-1}\delta_{\tilde{y}}\rangle_{\ell^2(\widetilde{G_N})}. \tag{17}$$

Theorem 12 (Anantharaman-Sabri [8])**.** *Assume that (G_N, W_N) satisfies* ***(BSCT)****,* ***(EXP)*** *and* ***(Green)****.*

Call $(\lambda_j^{(N)})_{j=1}^N$ the eigenvalues of $\mathcal{A}_N$ on $\ell^2(V_N)$, and let $(\phi_j^{(N)})_{j=1}^N$ be a corresponding orthonormal eigenbasis.

For each N, let $a = a_N$ be a function on V_N with $\sup_N\sup_{x\in V_N}|a_N(x)| \le 1$.

Then

$$\lim_{\eta_0\downarrow 0}\lim_{N\to+\infty}\frac{1}{N}\sum_{\lambda_j^{(N)}\in I_1}\left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\mu^N_{\lambda_j^{(N)}+i\eta_0}(x)\right| = 0\,,$$

for some family of probability measures μ_γ^N on V_N, indexed by a parameter $\gamma\in\mathbb{C}\setminus\mathbb{R}$, defined as follows :

$$\mu_\gamma^N(x) = \frac{\operatorname{Im}\tilde{g}_N^\gamma(\tilde{x},\tilde{x})}{\sum_{y\in V_N}\operatorname{Im}\tilde{g}_N^\gamma(\tilde{y},\tilde{y})}.$$

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Here, $\tilde{x} \in \widetilde{G_N}$ is a lift of $x \in V_N$.

Equivalently, for any $\epsilon > 0$, we have

$$\frac{1}{N}\left|\left\{\lambda_j^{(N)} \in I_1 : \left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\mu^N_{\lambda_j^{(N)}+i\eta_0}(x)\right| > \epsilon\right\}\right| \underset{N\to+\infty,\,\eta_0\downarrow 0}{\longrightarrow} 0\,. \tag{18}$$

Theorem 12 is not relevant unless we can compare the probability measures μ_γ^N with the uniform measure. A good test is to choose $a_N = \mathbb{1}_{\Lambda_N}$, the characteristic function of a set $\Lambda_N \subset V_N$ of size $\approx \alpha N$ for some $\alpha \in (0,1)$. In the special case where (G_N) is regular, the universal cover $\widetilde{G_N}$ does not depend on N (it is the $(q+1)$-regular tree); the Green function $\tilde{g}_N^\gamma(\tilde{x},\tilde{y})$ coincides with the limiting Green function $\mathcal{G}^\gamma(\tilde{x},\tilde{y})$ on the regular tree. Moreover, $\mathcal{G}^\gamma(\tilde{x},\tilde{x}) = \mathcal{G}^\gamma(o,o)$ for all $\tilde{x} \in \widetilde{G_N}$. It follows that μ_γ^N is the uniform probability measure on V_N (for every γ). So (18) implies that $\|\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx \alpha$ for most $\phi_j^{(N)}$. This shows that most $\phi_j^{(N)}$ are uniformly distributed, in the sense that if we consider any $\Lambda_N \subset V_N$ containing half the vertices, we find half the mass of $|\phi_j^{(N)}|^2$.

For general models, we cannot assert that $\mu_\gamma^N(\Lambda_N) = \alpha$ if $|\Lambda_N| \approx \alpha N$. Still, we prove that there exists $c_\alpha > 0$ such that for any $\Lambda_N \subset V_N$ with $|\Lambda_N| \geq \alpha N$, we have

$$\inf_{\eta_0\in(0,1)} \liminf_{N\longrightarrow\infty} \inf_{\lambda\in I} \mu^N_{\lambda+i\eta_0}(\Lambda_N) \geq 2c_\alpha\,. \tag{19}$$

Combined with (18), this implies

Corollary 13. *For any $\alpha \in (0,1)$, there exists $c_\alpha > 0$ such that for any $\Lambda_N \subset V_N$ with $|\Lambda_N| \geq \alpha N$, we have*

$$\frac{1}{N}\#\left\{\lambda_j^{(N)} \in I : \left\|\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\right\|^2 < c_\alpha\right\} \underset{N\longrightarrow+\infty}{\longrightarrow} 0\,.$$

Hence, while in the regular case we had $\|\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx \alpha$ for most $\phi_j^{(N)}$, in the general case, we can still assert that $\|\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \geq c_\alpha > 0$ for most $\phi_j^{(N)}$. This corollary indicates that our theorem can truly be interpreted as a delocalization theorem.

We also prove that that for any continuous $F : \mathbb{R} \to \mathbb{R}$, we have uniformly in $\lambda \in I$,

$$\frac{1}{N}\sum_{x\in V_N} F\left(N\mu^N_{\lambda+i\eta_0}(x)\right) \underset{N\longrightarrow+\infty}{\longrightarrow} \mathbb{E}\left(F\left(\frac{\operatorname{Im}\mathcal{G}^{\lambda+i\eta_0}(o,o)}{\mathbb{E}\left(\operatorname{Im}\mathcal{G}^{\lambda+i\eta_0}(o,o)\right)}\right)\right)\,. \tag{20}$$

This says that the empirical distribution of $\left(N\mu^N_{\lambda+i\eta_0}(x)\right)$ (when x is chosen uniformly at random in V_N) converges to the law of $\left(\frac{\operatorname{Im}\mathcal{G}^\gamma(o,o)}{\mathbb{E}(\operatorname{Im}\mathcal{G}^\gamma(o,o))}\right)$. This is a second way of saying that $\mu^N_{\lambda+i\eta_0}(x)$ is of order $1/N$: when multiplied by N, it has a non-trivial limiting distribution.

Remark 14. *The results proven in* [9] *actually hold for more general Schrödinger operators than adjacency matrices : one can consider weighted Laplacians (with conductances on the edges) and add a potential; in other words, on each G_N, we can consider a discrete Schrödinger operator $\mathcal{H}_N$. The limiting object in assumption **(BSCT)** is now a random*

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rooted tree $[\mathcal{T}, o]$ *endowed with a random Schrödinger operator* $\mathcal{H}$*. Assumption* ***(Green)*** *has to be modified, replacing the adjacency matrix* $\mathcal{A}$ *by the operator* $\mathcal{H}$*. Similarly, in the statement of the theorem, the Green functions* $\tilde{g}_N^\gamma$ *to be considered are those of* $\mathcal{H}_N$ *lifted to the universal cover* $\widetilde{G_N}$*.*

Remark 15. *In particular, our result applies to the case where the limiting system* $([\mathcal{T}, o], \mathcal{H})$ *is* $\mathcal{T} = \mathfrak{X}$ *(the* $(q+1)$*-regular tree) with an arbitrary origin* o*, and* $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{A} + \epsilon\mathcal{W}$ *where* $\mathcal{W}$ *is a random real-valued potential on* $\mathfrak{X}$*. More precisely the values* $\mathcal{W}(x)$ $(x \in \mathfrak{X})$ *are i.i.d. random variables of common law* ν*. This is known as the Anderson model on* $\mathfrak{X}$*. It was shown by A. Klein* [76] *that the spectrum of* $\mathcal{H}_\epsilon$ *is a.s. purely absolutely continuous on* $I = (-2\sqrt{q} + \delta, 2\sqrt{q} - \delta)$*, provided* ϵ *is small enough (depending on* δ*). This just assumes a second moment on* ν*. Under stronger regularity assumptions on* ν*, one can show that Assumption* ***(Green)*** *holds on* I *(see* [11]*, following Aizenman-Warzel* [2]*). Examples of sequences of expander regular graphs* G_N *with discrete Schrödinger operators* $\mathcal{H}_N$ *converging to* $([\mathfrak{X}, o], \mathcal{H}_\epsilon)$ *are given in* [10]*.*

Remark 16. *Examples of sequences of* non-regular graphs *satisfying our three assumptions were investigated in* [11]*. In the examples considered there, the limiting trees* $\mathcal{T}$ *are* trees of finite cone type*; roughly speaking, those are trees where the local geometry can only take a finite number of values. If* $\mathcal{A}$ *is the adjacency matrix of such a tree, we showed in* [11] *that the spectrum* σ *of* $\mathcal{A}$ *is a finite union of closed intervals, and that there are a finite number of points* $y_1, \ldots, y_\ell$ *in* σ *such that Assumption* ***(Green)*** *holds on any* I *of the form* $\sigma \setminus ([y_1 - \delta, y_1 + \delta] \cup \ldots \cup [y_\ell - \delta, y_\ell + \delta])$ *(for any* $\delta > 0$*). We showed – extending Remark 15 – that on such trees, Assumption* ***(Green)*** *remains true after adding a small random potential to* $\mathcal{A}_\mathcal{T}$*. Finally, we showed the existence of sequences* (G_N) *converging to* $\mathcal{T}$ *and satisfying the* ***(EXP)*** *condition.*

4. Perspectives and link with other work

4.1. **Random regular graphs.** It is important to stress the fact that Theorem 11 holds for *deterministic* sequences of graphs. For any sequence (G_N) satisfying the assumptions of the theorem, the conclusion holds for any observable a. As already noted, the result only says something about the delocalization of "most" eigenfunctions, where the "good" eigenfunctions exhibiting delocalization may depend on the choice of the observable a.

In the past years, there has been tremendous interest in spectral statistics and delocalization of eigenfunctions of *random* sequences of graphs and Schrödinger operators. Many papers consider *random* regular graphs, with degree going slowly to infinity [113, 45, 15, 13] or fixed [56, 14], sometimes adding a random i.i.d potential [56]. A (labelled) random regular graph on N vertices is produced as follows : given the vertex set $\{1, \ldots, N\}$, consider all the ways to draw edges between those vertices, that produce a $(q+1)$-regular graph (without self-loops and multiple edges); note that $(q+1)N$ has to be an even integer. Pick a graph at random for the uniform probability measure on all possible configurations.

The very impressive papers [15, 13, 14] show "quantum unique ergodicity" for the *adjacency matrix* of random regular graphs : given an observable $a_N : \{1, \ldots, N\} \longrightarrow \mathbb{R}$, for

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most $(q+1)$-regular graphs on the vertices $\{1, \dots, N\}$ we have that $\sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2$ is close to $\langle a_N \rangle$ for *all* indices j, with an excellent control of the remainder term :

Theorem 17 (Bauerschmidt-Huang-Yau [14])**.** *Let ω be such that $\sqrt{q} \geq (\omega+1)2^{2\omega+45}$.*

(i) With probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph,

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions associated to eigenvalues such that $|\lambda_j \pm 2\sqrt{q}| > (\log N)^{-3/2}$.

(ii) (Quantum Unique Ergodicity for random regular graphs) Given an observable $a_N : \{1, \dots, N\} \longrightarrow \mathbb{R}$, we have, with probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph, for N large enough,

$$\left| \sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2 - \langle a_N \rangle \right| \leq \frac{(\log N)^{250}}{N} \sqrt{\sum_x |a_N(x)|^2}, \tag{21}$$

for all eigenfunctions associated to eigenvalues $\lambda_j \in (-2\sqrt{q}+\epsilon, 2\sqrt{q}-\epsilon)$ (bulk eigenvalues).

In particular, if $a_N = \mathbb{1}_{\Lambda_N}$ where $\Lambda_N \subset \{1, \dots, N\}$, we find

$$\left| \sum_{x \in \Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N} \right| \leq \frac{(\log N)^{250}}{N} \sqrt{|\Lambda_N|}.$$

(Note in passing that $\omega > 8$ implies that $q > 2^{128}$).

So the ℓ^∞-norm of bulk eigenfunctions is as small as can be, and QUE takes place on sets of size $|\Lambda_N| > (\log N)^{500}$. By comparison, in Theorem 11, for graphs whose girth goes to ∞, our proof would never do better than

$$\left| \sum_{x \in \Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N} \right| \leq \frac{1}{\sqrt{N \log N}} \sqrt{|\Lambda_N|}.$$

So Theorem 17 is a considerable strengthening of (18), that only said something for *most* indices j and for $|\Lambda_N| > N^{1/2}$. This possibility to prove QUE is, of course, due to the fact that a_N is probabilistically independent of the choice of the graph; in Theorem 11 and (18), a_N could depend on the graph. It might well be that a positive proportion of graphs contradicts QUE, if we were allowed to choose observables a_N depending on the graph (this is a completely open question). Note also that if we are given a deterministic sequence of regular graphs (for instance, say, the Lubotzky-Phillips-Sarnak Ramanujan graphs [86]), we do not know if Theorem 17 applies to it, as it is an *almost sure* conclusion.

Remark 18. *Note that we emphasized Theorem 17 from* [14] *because our main concern here is the delocalization of eigenfunctions. The main focus of* [14] *is however on the universality of the local spectral statistics for random regular graphs. This would deserve a separate paper.*

The recent paper by Backhausz and Szegedy [12] proves a very important result, saying that for almost all random regular graphs G_N on N vertices, and *all* eigenvectors $\phi_j^{(N)}$s,

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the value distribution of $\sqrt{N}\phi_j(x)$ as x runs over $\{1, \ldots, N\}$ is close to some Gaussian $\mathcal{N}(0, \sigma_j^2)$ with $0 \leq \sigma_j \leq 1$. More generally, for any $R \geq 0$, picking x uniformly at random in $\{1, \ldots, N\}$ and looking at the values of $(\sqrt{N}\phi_j(y))_{y, d(y,x) \leq R}$ in the vicinity of x in G_N, the obtained random function is close in distribution to a gaussian process on $B_{\mathfrak{X}}(o, R)$. The covariance function has to be of the form $\sigma_j^2 \Phi_{\lambda_j}(d(x, y))$ where Φ_{λ_j} is the spherical function of parameter λ_j on the $(q + 1)$-regular tree $\mathfrak{X}$. Proving that $\sigma_j = 1$, or even just that $\sigma_j \neq 0$, is a challenge; it would amount to proving that eigenfunctions cannot be localized on $o(N)$ vertices. Theorem 10 does not say this, it only says that eigenfunctions cannot be localized on N^α vertices. Our Theorem 11, or the random version Theorem 17 do not say this either, because we can only test one observable a_N at a time. The indices j for which (18) holds, or the set of graphs satisfying (21), depend on a_N. If we wanted to have a common set that does the job for all observables (whose number is exponential in N), we would need to have exponential error bounds in (18) or (21).

4.2. **From graph Laplacians to Hecke operators.** What do these discrete results teach us about the problem we were originally interested in, namely the eigenfunctions of the Laplace-Beltrami operators on a riemannian manifold ? A natural question that comes to mind is to try to adapt Theorem 11 to sequences of graphs that are finer and finer triangulations of some given Riemann surface. With the appropriate choice of conductances on the edges, the corresponding discrete Laplacians approximate the continuous Laplacian. At the present time, we are unable to say anything about such graphs, because they have many loops and this is excluded by the hypotheses of Theorem 11.

But let us throw a look in a different direction, that of "Arithmetic quantum ergodicity".

Consider $\mathbb{S}^2$, the 2-dimensional sphere with its usual, round metric. The eigenfunctions of $\Delta_{\mathbb{S}^2}$ are spherical harmonics, i.e. restrictions to $\mathbb{S}^2 \subset \mathbb{R}^3$ of harmonic homogeneous polynomials in 3 variables. Harmonic homogeneous polynomials of degree ℓ give rise to eigenfunctions of $\Delta_{\mathbb{S}^2}$ for the eigenvalue $-\ell(\ell+1)$ (the dimension of the eigenspace is $2\ell+1$).

The Laplacian $\Delta_{\mathbb{S}^2}$ commutes with the infinitesimal rotation $\mathbf{J}_{12} = \frac{1}{i}\left(x_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$. Note that J_{12} is a differential operator of order 1, and that its principal symbol is the kinetic momentum around the vertical axis.

The basis $(\phi_n) = (Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ of joint eigenfunctions of $\Delta_{\mathbb{S}^d}$ and J_{12} cannot satisfy the conclusions of Theorem 5. In fact, using the same notation as in §2.4, let us consider a subsequence (ϕ_{n_k}) such that $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converges for all A; the limit is of the form $\int_{S^*M} \sigma^0(A) d\mu$, where μ is a probability measure on S^*M. The fact that ϕ_{n_k} is an eigenfunction of J_{12} is converted into the property that μ is carried by a level set of the kinetic momentum (which is a submanifold of positive codimension in S^*M); thus μ cannot be the Lebesgue measure.

Because the spectrum of the Laplacian has huge multiplicities, one can wonder whether other bases of eigenfunctions on the sphere satisfy Theorem 5. Zelditch had the idea of considering random eigenbases [118]. He showed that "almost every" choice of eigenbasis satisfies Theorem 5 (this was strengthened to Quantum Unique Ergodicity by Van Der Kam [114]).

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Brooks, Le Masson and Lindenstrauss [37] showed quantum ergodicity for an explicit basis of eigenfunctions of $\Delta_{\mathbb{S}^2}$, that are also eigenfunctions of a kind of "discrete" Laplacian on $\mathbb{S}^2$: for $g_1, \ldots, g_k$ a finite set of rotations in $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} (f(g_j x) + f(g_j^{-1} x))$$

commutes with $\Delta_{\mathbb{S}^2}$.

Theorem 19 (Brooks, Le Masson and Lindenstrauss [37]). *Assume that $g_1, \ldots, g_k$ generate a free subgroup of $SO(3)$.*

For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an orthonormal family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k.

Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1} \sum_{j=1}^{2\ell+1} \left| \int_M a(x) |\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\ell \longrightarrow \infty}{\longrightarrow} 0.$$

Restricting T_k to the space of spherical harmonics of degree ℓ is shown to be roughly the same as letting T_k act on a discretization of the sphere by an ℓ^{-1}-net. This is the same as studying the Laplacian on a $2k$-regular graph with $N \sim \ell^2$ vertices, and if $g_1, \ldots, g_k$ generate a free subgroup, this graph has few short loops. Thus, Theorem 19 is similar to Theorem 11. The theorem on regular graphs can serve as a canvas to prove Theorem 19.

Remark 20. *T_k is not a pseudodifferential operator, so the argument sketched above to show that the basis $(Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ could not satisfy quantum ergodicity does not apply here.*

Remark 21. *We note that for very special choices of rotations – rotations that correspond to norm n elements in an order in a quaternion division algebra, the operators T_k are called Hecke operators. It has been conjectured by Böcherer, Sarnak, and Schulze-Pillot* [25] *that such joint eigenfunctions satisfy the much stronger quantum unique ergodicity property. This conjecture is still open.*

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The idea of adapting a result on discrete graphs to the realm of Hecke operators on arithmetic manifolds had already been used in [39]. In 2000, Bourgain and Lindenstrauss had considered the measures μ obtained in Theorem 7, when the eigenfunctions (ϕ_n) are joint eigenfunctions of Δ and of the *infinite* family of Hecke operators on an arithmetic hyperbolic surface (e.g., the modular surface). They were able to show that μ has positive entropy on *almost-every ergodic component*, and this fact was a key ingredient in the proof of Arithmetic Quantum Unique Ergodicity by Lindenstrauss [84]. In [39], Brooks and Lindenstrauss used the fact that a Hecke operator, restricted to a net in the manifold M, acts similarly to the discrete Laplacian on a regular graph with few short loops, to adapt Theorem 10 and show that μ has positive entropy on almost-every ergodic component, using only *one* Hecke operator.

In the next paragraph, we mention another continuous adaptation of Theorem 11 : instead of thinking of discrete Laplacians living in a riemannian manifold and restricted to a finer and finer net, we look at riemannian manifolds that get larger and larger :

4.3. **Quantum ergodicity on Riemann surfaces of high genus.** Theorems 11 and 12 were dubbed as "quantum ergodicity" theorems in reference to the historical Theorem 5. However, we already noted a difference in the meaning of these results. Theorem 5 holds in the high-frequency régime, whereas the graph-results deal with the large-scale régime. So, a continuous analogue of Theorem 11 would be to consider compact Riemannian manifolds whose volume goes to infinity. Such a result was obtained by Le Masson and Sahlsten for Riemann surfaces of high genus :

Theorem 22 (Le Masson– Sahlsten [80])**.** *Let (S_N) be a sequence of hyperbolic surfaces, whose genus (equivalently, volume) goes to ∞.*

(EXP) *Assume the first eigenvalue $\lambda_1(N)$ of $-\Delta$ on S_N is bounded away from 0 as $N \longrightarrow \infty$.*

(BSH)*Assume there are few short geodesics; in other words, (S_N) converges in the Benjamini-Schramm sense to the hyperbolic disc : for any $R > 0$,*

$$\lim_{N\longrightarrow+\infty} \frac{\mathrm{Vol}\{x \in S_N, \rho(x) < R\}}{\mathrm{Vol}(S_N)} = 0$$

where $\rho(x)$ means the injectivity radius at x.

Fix an interval $I \subset (1/4, +\infty)$.

Let $(\phi_i^{(N)})$ be an orthonormal basis of eigenfunctions of the Laplacian on S_N.

Let $a = a_N : S_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in S_N$. Then

$$\lim_{N\longrightarrow+\infty} \frac{1}{Vol(S_N)} \sum_{\lambda_i(N)\in I} \left| \int_{S_N} a(x)|\phi_i^{(N)}(x)|^2 dx - \langle a\rangle \right|^2 = 0$$

where $\langle a\rangle = \frac{1}{Vol(S_N)} \int_{S_N} a(x)dx$.

We note that $(1/4, +\infty)$ is the L^2-spectrum of the Laplacian on the hyperbolic disc. This spectrum is purely absolutely continuous. So, like in the graph case, we are working with the sequence of compact S_N converging to an infinite-volume simply connected manifold, with purely absolutely continuous spectrum. It would be interesting to find more examples of such manifolds (and to extend Theorem 22 to that more general setting), but we have already mentioned in §2.5.1 the difficulty of proving absolutely continuous spectrum.

A tremendously interesting question would to put this result in the framework of *random* Riemann surfaces :

- does Quantum Unique Ergodicity hold for large *random* Riemann surfaces, in the spirit of Theorem 17 ?
- for a typical random Riemann surface, is the value distribution of the eigenfunctions $(\phi_i^{(N)})$ asymptotically gaussian, similarly to the case of random regular graphs recently treated by Backhausz-Szegedy in [12] ? This would come very close to

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justifying Berry's Random Wave ansatz [20] – the latter was formulated in the high-frequency régime, but a version in the large-scale limit would also be of high interest.

The most natural notion of random Riemann surface of genus g is obtained by putting the Weil-Petersson volume measure on their moduli spaces. The volume of the moduli space of Riemann surface of genus g was computed by Mirzakhani (see [91, 92] and references therein), and she could give its asymptotic behaviour as $g \longrightarrow +\infty$. She showed that a random Riemann surface has a uniform spectral gap in the spectrum of the Laplacian, as $g \longrightarrow +\infty$; this is similar to what is known for random regular graphs. She also obtained asymptotic information about the law of the length of the shortest closed geodesic and the shortest separating geodesic. However, this model of random Riemann surfaces does not seem flexible enough to allow for a direct transposition of the wonderful result of [12]. This is a very intriguing topic to explore.

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